A Continuous-Response Task with Nondeterminate, Contingent Reinforcement

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An experiment dealing with a continuous-response task with nondeterminate contingent reinforcement is analyzed in detail. The subject's task was to attempt to hit a target located at a variable position on the circumference of a circle. The subject could not see this target but was simply told after each response whether he had hit or missed the target. The sixty subjects were randomly assigned to one of two groups, the only difference in the groups being the size of the target. The asymptotic response data are analyzed with respect to three different one-element models. Only one of these three models, the identity model, is at all satisfactory. For the analysis of sequential data this model is extended to an N-element version.

In this experiment subjects are instructed to "hit" an unseen target which is said to be located at some point on the circumference of a circle. If the subject's response lies within a specified distance of the target, he is informed that he has a hit on that trial, otherwise a miss. The nondeterminate aspect of this experiment refers to the fact that the subject is not informed of the exact location of the target after a miss.

The theoretical background for this experiment has been presented in detail in Suppes and Zinnes (1961). It will suffice to summarize briefly the theoretical assumptions and results that are useful here. Although both "linear" and "stimulus-sampling" versions of the theory are developed in Suppes and Zinnes, in this paper we have limited ourselves to the stimulus-sampling version. Predictions based on the linear theory tend to be either equivalent to a corresponding stimulus-sampling model or they are mathematically difficult to work with because closed-form expressions are not obtainable.

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In the stimulus-sampling version of the theory it is assumed that on each trial the state of the organism is represented by a symmetric density function $k(x, z)$, where z is the mean of the distribution and x is a value of the response random variable. If on trial n, $z = z_n$, then the probability that the response on this trial will lie between a and b is given by

$$\int_a^b k(x, z_n)\, dx.$$

In the present experiment, after the subject has made x_n on trial n, i.e., selected a point on the response continuum, one of two events occurs: the subject is either informed that he has "hit" the "target" or that he has "missed" it. When a hit occurs it is assumed that with probability θ, $z_{n+1} = x_n$ and with probability $1 - \theta$, $z_{n+1} = z_n$; that is, the mean of distribution $k(x, z)$ either shifts to the reinforced response x_n on the following trial or it stays put. When a miss occurs three alternative assumptions are considered. With probability θ, one of the following obtains:

$$(I) \qquad z_{n+1} = z_n \qquad \text{(Identity Model)},$$

$$(S) \qquad z_{n+1} = x_n + \pi \qquad \text{(Symmetric Model)},$$

$$(U) \qquad j(z_{n+1}) = \frac{1}{2\pi} \qquad \text{(Uniform Model)},$$

where $j(z_{n+1})$ is a density function.

In all cases when a miss occurs, then with probability $(1 - \theta)$, $z_{n+1} = z_n$. The constant π in (S) and (U) occurs because of the circular nature of the response continuum used in the present experiment. The identity model assumes that on nonreinforced trials, i.e., trials in which the subject is informed he has missed the target, the mean of distribution $k(x, z_m)$ does not change. The symmetric model, defined by (S), assumes that on nonreinforced trials when the mean of the distribution $k(x, z)$ changes it goes to a point diagonally opposite the response x. In the uniform model (U) it is equally likely to go to any point on the circular continuum.

In Suppes and Zinnes (1961) expressions for asymptotic densities and conditional densities are derived for each of these alternative models. The more useful expressions and the method of estimation used will be given in the appropriate parts of the result section.

Some of the findings reported here have made it necessary to generalize the above theory to N elements. Detailed derivations of the relevant results are given in the Appendix. The general notions involve associating a separate $k(x, z)$ distribution with each element. The response depends on which element (i.e., distribution) is sampled on a trial and all the above assumptions then apply to the mean of the sampled distribution.

METHOD

Subjects. The subjects were 60 undergraduate students at Stanford University. They were paid for their participation in the one-hour experimental session.

Apparatus. The apparatus is described in detail in Suppes and Frankmann (1961). It consists of a large translucent screen with a knob in the center. Turning the knob rotates a narrow slit of light around a 5-ft-diameter circle. Behind the screen is a 400-division scale and a pointer that is used to determine the position of the light at each point of its circular path. The scale itself can be rotated so that the zero point is located at different positions of the circle. In the present experiment chimes were used to indicate one of the two outcomes (a "hit" or a "miss") on each trial. The "hit" chime consisted of one tone and the "miss" chime two tones. A sign in front of the subjects served to remind them which chime was which.

Procedure. The essential part of the instructions read to the subjects were as follows:

"In this experiment we are studying how people learn to locate a moving target. Your task will be to try to predict the location of a target on the screen before you. You can think of this as trying to hit the target. After each guess you will be informed whether you have hit the target or missed the target. However, you will never actually see the target. Although the target may move from trial to trial, there will always be one target at some place on every trial.

"If you have hit the target you will hear the following chime 'hit.' However, if you have missed you will hear instead this chime 'miss.' The actual size of the target in this experiment projects an angle of 72 degrees (121 degrees). At first you will have to guess about the target locations, but with practice your predictions should improve. Try to get as many hits as you possibly can and remember on each trial the center of the target can be *at any* point on the circle."

Design. For all subjects the location of the center of the target on each trial was determined by randomly sampling from the interval $(-\pi, \pi)$ with the probability density function

$$f(y) = \frac{4}{3\pi} \cos^4 \left(\frac{y}{2}\right).$$

(The actual sampling was done using the Rand Deck of random numbers and the cumulative density function corresponding to the above density function.) In the range $-\pi \leqslant y \leqslant \pi$ this density function is unimodal and symmetric around $y = 0$, and in fact it is not very different from a normal distribution. This distribution was used instead of the normal distribution primarily because its cyclical properties reflect in a natural way the periodic continuum used in this experiment, and therefore many of the mathematical derivations are simplified.

The width or size of the target differed for two groups. For half of the subjects (Group S) the target angle was 80 units on the 400-division scale used ($2\pi/5$ radians) and for the other half (Group L) the target angle was 134 units ($.67\pi$ radians). Thus Group S subjects were informed by the chimes that they had a "hit" whenever their responses did not deviate more than 40 units from the value of y sampled on that trial, and for Group L whenever their responses did not deviate more than 67 units from y. It should be noted that although the value of y sampled on each trial is independent of the subject's response, the actual reinforcing event (a "hit" or a "miss") that occurs on each trial is highly dependent (contingent) on his response.

A rough idea can be obtained about the reinforcement schedule experienced by the subjects by noting that for Group S subjects the probability of a hit for a response at the center of the distribution ($y = 0$) is equal to

$$\frac{4}{3\pi}\int_{-\pi/5}^{\pi/5} \cos^4 \frac{y}{2}\, dy = .50,$$

and for Group L it is

$$\frac{4}{3\pi}\int_{-.335\pi}^{.335\pi} \cos^4 \frac{y}{2}\, dy = .75.$$

This means that Group S subjects can be expected to obtain fewer than 50% hits and Group L fewer than 75% hits, since a response at the center of the reinforcement distribution is the optimum response for this task.

The actual sequence of y values used was different for each subject within each group, but matching subjects across the two groups received the same sequence. The zero position of the scale also differed for each subject within a group but it remained the same for matched subjects across groups. It should be noted that the net effect of changing the physical location of the zero point of the scale is to change the physical location of the mean position of the target center, and therefore the optimal response for each subject within a group is at a different point of the circle. Each subject received 300 trials with a brief rest period introduced after the first 200 trials. The total experimental time following the reading of instructions was about 45 minutes.

RESULTS AND DISCUSSION

Asymptotic response histogram. Some of the general properties of the performance of Groups S and L are indicated in Table 1, which shows that the number of hits increases over blocks of trials for both groups and that the response variance decreases over trial blocks.

TABLE 1

OBSERVED MEANS AND VARIANCES OF RESPONSES IN FIFTY-TRIAL BLOCKS

	Group S			Group L		
Block	Mean Prop. of Hits	Response Mean	Response Variance	Mean Prop. of Hits	Response Mean	Response Variance
1	.264	−.0107	2.54	.411	−.0528	2.46
2	.297	−.0478	2.05	.455	−.0408	1.93
3	.278	.0320	1.97	.500	.0242	1.55
4	.313	−.0371	1.62	.542	−.0185	1.43
5	.345	.0050	1.44	.544	−.0521	1.28
6	.350	−.0138	1.32	.556	−.0399	1.18

Note. Response mean and variance given in radians (1 degree = .017 radians).

Both facts indicate that the two groups of subjects continuously improved their performance throughout the duration of the experiment and therefore did not reach an asymptotic level of responding. However, both groups show some signs of leveling off over the last three or four blocks of trials. In the following analysis, therefore, we shall make the assumption that the responses during the last three blocks of trials for Group S and the last four blocks of trials for Group L are sufficiently near asymptote to produce negligible errors when asymptotic expressions are utilized. This assumption is perhaps the most tenuous of any we have introduced for purposes of data analysis.

In Table 1 it may also be noted that the mean response for both groups is quite close to zero over all blocks of trials. Since the continuum is circular and the physical location of the mean of the target positions (for convenience, referred to here as the reinforcement distribution) is changed from subject to subject, this result merely shows that the counterbalancing effectively averaged out possible spatial biases. It does not indicate that the subjects can locate the mean of the reinforcement distribution [$E(y)$] on the first block of trials.

The means and variances of the "asymptotic" response histograms for the individual subjects of groups S and L are reported in Table 2.

TABLE 2

MEAN AND VARIANCE OF THE RESPONSE HISTOGRAMS OF SUBJECTS BASED ON THE LAST 150 TRIALS

Subject	Group S		Group L		Subject	Group S		Group L	
	Mean	Variance	Mean	Variance		Mean	Variance	Mean	Variance
1	−.080	2.19	−.176	1.32	16	−.220	1.59	−.270	.71
2	.380	1.50	.003	1.62	17	−.238	.19	−.185	.86
3	.064	1.16	.157	1.78	18	.041	1.57	−.170	2.32
4	.119	1.39	.094	2.32	19	−.163	1.42	−.166	2.50
5	−.075	.88	.000	.88	20	−.132	2.16	.104	1.32
6	.232	1.12	−.204	3.16	21	.035	1.45	.151	.74
7	−.301	1.11	−.082	1.03	22	−.122	1.35	.009	.59
8	.223	2.00	.054	.57	23	.170	1.21	.016	.40
9	.228	1.45	.403	.84	24	.311	.43	−.273	.93
10	−.251	1.54	−.101	.45	25	.044	3.03	−.022	.91
11	−.069	.41	.050	.98	26	−.320	.68	−.047	1.58
12	.321	.98	−.252	.72	27	−.113	2.14	.060	1.41
13	−.148	1.29	.204	2.12	28	.122	3.47	.179	2.72
14	−.116	.09	.280	.18	29	−.270	.89	−.233	.52
15	−.298	.85	.094	1.70	30	.144	3.13	−.321	2.76
Average	−.0154	1.464	−.0217	1.362					

It may be noted that with very few exceptions, the variances of these histograms are not close to zero. The exceptions are subjects 14 and 17 in Group S with a variance of .09 and .19, respectively, and subject 14 in Group L with a variance of .18. The individual subjects by and large do not show any strong tendency either to optimize their responses or to respond consistently at some point away from $E(y)$. Comparing the response variance to $\mathrm{var}(y)$, as is usually done in the determinate case, is not too meaningful here, since the experimental outcomes depend heavily on the size of the target. In the extreme case when the target is equal to 2π, the subject will have a hit on each trial independent of $\mathrm{var}(y)$, and at the other extreme if the target is indefinitely small, the subject will have a miss on each trial. The reason why the mean responses of some individual subjects differ from $E(y)$, seems to be due to the strong pull toward one of the two horizontal or vertical points (i.e., the points at 3, 6, 9, and 12 o'clock), and this is particularly noticeable when $E(y)$ for a given subject falls near one of these four positions.

The asymptotic group histograms are plotted in Figs. 1 and 2. For the three models described in the introduction, this histogram

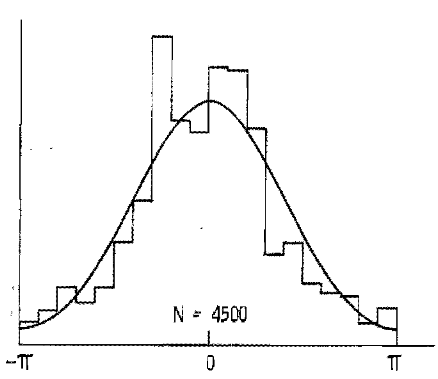

FIG. 1. Theoretical and observed asymptotic response distribution for Group S. Theoretical predictions are based on the identity model.

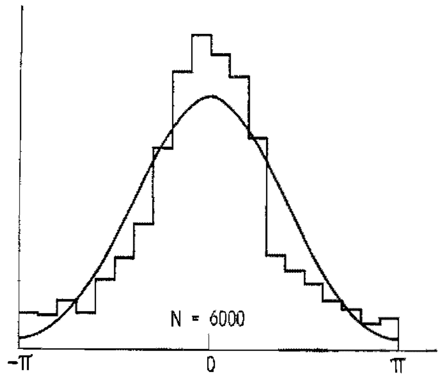

FIG. 2. Theoretical and observed asymptotic response distribution for Group L. Theoretical predictions are based on the identity model.

depends on the nature of the distribution $k(x, z)$, and not on the learning-rate parameter θ. The family of functions selected for $k(x, z)$ is

$$k_l(x, z) = C_l \cos^{2l}\left(\frac{x - z}{2}\right) \tag{1}$$

where C_l, the normalizing factor, is given recursively by

$$C_l = \frac{2l}{2l - 1} C_{l-1}$$
$$C_1 = \frac{1}{\pi}. \qquad (2)$$

The parameter l determines the variance of the distribution $k(x, z)$ and it is this parameter which needs to be estimated for all of the three models. Table 3 shows the results of the attempt to fit the theoretical variance to the obtained variance. For models S and U the situation, quite surprisingly, is hopeless. Neither can be made to fit the observed data. In both cases the theoretical variance decreases with l, but all too quickly levels off at a point considerably larger than the obtained variance. The obtained variances, as indicated in Table 2, for the S and L groups are 1.464 and 1.362, respectively. Not all of the cells of Table 3 are filled in because of the lengthy computations involved. The computations that were made for these two models, along with some general inspection of the various functions involved, appeared sufficient to justify abandoning these models at the outset. The observed variance is simply too small for either one of the two models to cope with. It is unlikely that another class of functions for $k(x, z)$ would alter this fact. Analyses carried out in Suppes and Frankmann (1961) based on two very different distributions, the uniform and the symmetric beta distribution,

TABLE 3

THEORETICAL PREDICTIONS OF THE ASYMPTOTIC RESPONSE VARIANCE FOR DIFFERENT $k_l(x, z)$ DISTRIBUTIONS[a]

Type of theory	Value of l									
	1	2	3	4	5	6	7	8	9	10
	Group S									
Identity	2.043	1.648	1.457	—	—	—	—	—	—	—
Symmetric	2.899	2.803	2.760	2.735	2.719	2.707	—	—	—	—
Uniform	3.009	—	—	—	2.788	—	—	—	—	—
	Group L									
Identity	—	1.831	1.658	1.555	1.487	1.439	1.403	1.375	1.353	1.335
Uniform	—	—	—	—	2.418	—	—	—	—	—

[a] $k_l(x, z)$ is defined in Eq. 1.

resulted in nearly indistinguishable asymptotic response distributions. In general, it appears that the theoretical predictions are relatively insensitive to the specific form of $k(x, z)$. (Since these two models were abandoned early in the analysis it does not appear useful to summarize here any of the relevant asymptotic formulas and computational methods used.)

There remained the identity model, which is analytically the most difficult to work with. A long (and sad) story could be told about various attempts to simplify this model or find a reasonable approximation to it. It will suffice to say that in most cases numerical methods performed by high-speed computers rather than analytical techniques were finally employed. The asymptotic-response density $r(x)$ for the identity model is given by the integral equation (see Suppes and Zinnes, 1961, Eq. 48)

$$r(x) = \int_{-\pi}^{\pi} k(x, z)H(z)r(z)\, dz, \tag{3}$$

where, omitting obvious limits of integration,

$$H(z) = \frac{\pi_1(z)}{\int \pi_1(x)k(x, z)\, dx} \tag{4}$$

and $\pi_1(x)$, the probability that a response at x will terminate in a hit, is given by

$$\pi_1(x) = \int_{\pi-\alpha}^{x+\alpha} f(y)\, dy. \tag{5}$$

Substituting the "reinforcement" density

$$f(y) = \frac{4}{3\pi} \cos^4 \left(\frac{y}{2}\right) \tag{6}$$

into (5) yields

$$\pi_1(x) = \frac{\alpha}{\pi} + \frac{4}{3\pi} \sin \alpha \cos x + \frac{1}{6\pi} \sin 2\alpha \cos 2x \tag{7}$$

where the values of α for Groups S and L are $\pi/5$ and $67\pi/200$, respectively.

For Group S it was possible to solve (3) analytically for $r(x)$ for each value of l. From Table 3 it will be seen that the asymptotic response variance for the case $l = 3$ is 1.457, which approximately matches the observed variance of 1.464. For this case, the asymptotic-response density is given by the numerical equation

$$r(x) = \frac{1}{2\pi} + (.1488875) \cos x + (.0120453) \cos 2x + (.0000002) \cos 3x. \tag{8}$$

The smooth curve in Fig. 1 was obtained by evaluating the above function for 20 equally-spaced values of x.

Group L data were analyzed entirely by numerical methods. The integral in (3) was replaced with a finite sum, thus converting the integral equation into the set of linear simultaneous equations

$$r(x_j) = \sum_{i=1}^{20} k_l(x_j, z_i) H(z_i) r(z_i), \qquad j = 1 \ldots 20. \tag{9}$$

The values of x_j and z_i in (9) are just the twenty equally-spaced values in the interval $(-\pi, \pi)$. This set of 20 linear simultaneous equations in the unknown $r(x_j)$ was solved for each value of l. [Actually due to the symmetry properties of the functions entering into (9) it was sufficient to solve 10 of the equations for 10 of the unknown $r(x_j)$.]

The variance of the theoretical asymptotic histogram determined by these 20 values of $r(x_j)$ is given in Table 3 for each one of the l values. This table shows that when $l = 9$ the theoretical variance of the Group L asymptotic response distribution of 1.353 matches the obtained variance of 1.362. Thus the 20 theoretical points plotted in Fig. 2 correspond to the 20 values of $r(x_j)$ obtained when $l = 9$.

The remaining analysis we performed is based on values of $l = 3$ for Group S and $l = 9$ for Group L. The results would have been much more satisfying if $k(x, z)$ had shown some invariance over reinforcement conditions, that is, if the values of l had turned out to be equal for both groups, but such is definitely not the case. The distributions k_3 and k_9 have quite different variances. Another way of describing the same limitation of the identity model in accounting for these data is to note that for a fixed $k(x, z)$ distribution, the identity model requires the response variance to increase with target angle. (See Suppes and Zinnes, 1961, p. 386.) Since the response variance for Group L is smaller, not larger, than Group S, it is necessary to use different $k(x, z)$ distributions for these two groups.

For purposes of performing a goodness-of-fit test, the asymptotic-response histogram based on every third response was used. (The lack of independence of the observations under the present Markovian assumption means that the usual chi-square statistic based on all asymptotic responses would not have a chi-square distribution. This issue is discussed in more detail in Suppes and Atkinson, 1960.) The chi square for the Group S and L histograms are 138 and 198, respectively. In both cases the degrees of freedom are 18, which means that the two chi-square terms are clearly too large. Although the magnitude of these two chi-square terms could have been reduced somewhat if more efficient estimation procedures had been used, the large values found here are typical of those generally found in learning studies in which the number of observations is quite large and the power of the chi-square test is close to one.

Asymptotic distribution of responses following a miss. The histograms in Figs. 3 and 4 are based on responses that were preceded by a miss.

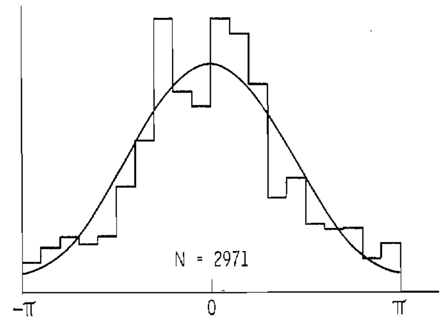

FIG. 3. Theoretical and observed asymptotic conditional response distribution, $\lim_{n\to\infty} r(x_{n-1} \mid Y_{0,n})$, for Group S.

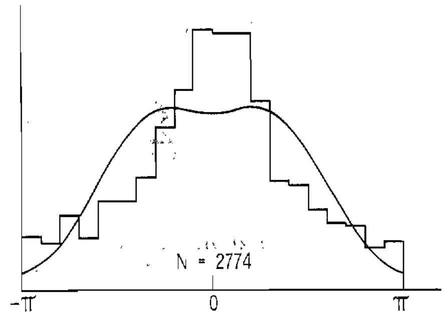

FIG. 4. Theoretical and observed asymptotic conditional response distribution, $\lim_{n\to\infty} r(x_{n+1} \mid Y_{0,n})$, for Group L.

The relevant theoretical expression for the identity model (see Eq. 61 in Suppes and Zinnes, 1961) is

$$\lim_{n\to\infty} r_{n+1}(x \mid Y_{0,n}) = \frac{r(x) - \int k(x, z)\pi_1(z)r(z)\, dz}{\int \pi_0(x)r(x)\, dx} \tag{10}$$

where $\pi_0(x) = 1 - \pi_1(x)$ and $Y_{0,n}$ indicates a miss on trial n. The two integrals in the above equation were evaluated by using numerical methods based on the 20 equally-spaced values of $r(x)$ evaluated previously.

TABLE 4

COMPARISON OF THEORETICAL AND OBSERVED ASYMPTOTIC VARIANCES FOR TWO TYPES OF DISTRIBUTIONS AND GOODNESS-OF-FIT OF THE THEORETICAL DISTRIBUTIONS

	Group S		Group L	
Variance	Var. of asymp. responses	Var. of asymp. responses following a miss	Var. of asymp. responses	Var. of asymp. responses following a miss
Theoretical	1.46	1.59	1.35	1.86
Observed	1.46	1.69	1.36	1.86
Chi square	138.	135.	198.	189.
df	18	19	18	19

The results of these calculations are indicated by the smooth lines in Figs. 3 and 4. It should be emphasized that the theoretical density functions of (10) contain no new free parameters to be estimated. It is encouraging therefore that the theoretical variances for both groups given in Table 4 agree quite well with the observed variance. Furthermore, the fact that the variance of the theoretical conditional distribution $r(x_{n-1} \mid Y_{0,n})$ is larger than the variance of the (nonconditional) distribution $r(x)$ is reflected in the data for both groups as well. This latter point is a significant prediction of the identity theory, since it is by no means obvious whether the conditional distribution should have a larger or smaller variance than the nonconditional distribution.

The chi squares based on nonconsecutive responses that were preceded by a miss (given in Table 4) are large (135 and 189 for Groups S and L, respectively), but they are no larger than the chi squares of the nonconditional distributions, although the degrees of freedom have increased slightly (from 18 to 19).

Although the theoretical and observed variances of the conditional distributions agree quite well, a closer look at the histogram in Fig. 4 is less encouraging. The theoretical distribution quite unexpectedly has a slight bimodal character, a property which is not reflected to any degree in the observed histogram. One way of altering this aspect of the theoretical distribution is to introduce more stimulus elements into the identity theory, but we reserve this approach for the more detailed sequential analysis to be considered next.

Asymptotic transitions following a miss. A more detailed examination of the responses following a miss is afforded by noting what the response was when the miss occurred, or in theoretical terms, by considering

$$\lim_{n\to\infty} r(x_{n-1} \mid a \leqslant x_n \leqslant b,\ Y_{0,n})$$

for various values of a and b. The relevant data are plotted in Figs. 5 and 6. In these figures the response interval has been broken into 10 equal intervals. For interval 1, for example, $a = -\pi$ and $b = -4\pi/5$.

The sparsity of data in some of the cells made it desirable to collapse the 10×10 transition table to a 5×10 table by combining cells (i, j) and $(11 - i,\ 11 - j)$. The symmetry properties of the circle and all relevant theoretical functions make this collapsing feasible.

It will be noted in Figs. 5 and 6 that the upper histograms are based on relatively few observations. For example, there are about three times as many observations in the fifth histogram of Fig. 5 as there are in the first histogram of this table. In Fig. 6 the number of cases for these two histograms differs by a factor of four. Therefore, it should be realized that the histograms in these two figures have different degrees of reliability.

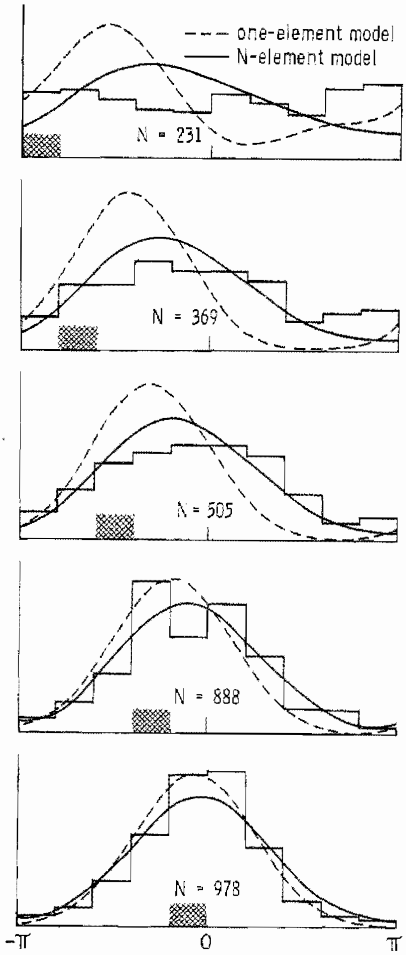

FIG. 5. Theoretical and observed asymptotic transitional distributions $\lim_{n\to\infty} r(x_{n+1} \mid a \leq x_n \leq b, Y_{0,n})$ for Group S. The interval (a, b) for each of the above histograms is indicated by a darkened section on the abscissa of the histogram. The value of N indicated in each histogram indicates the number of cases on which that histogram is based.

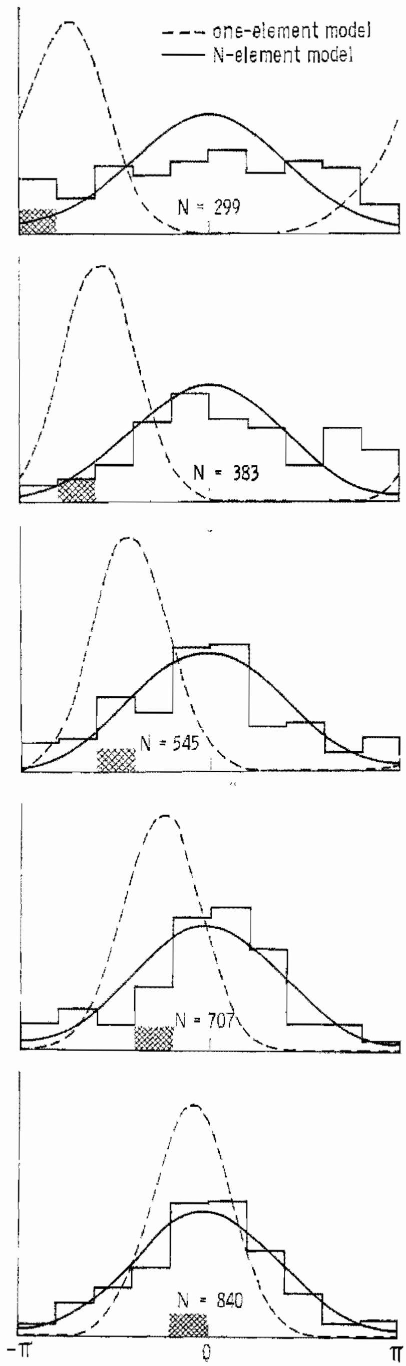

FIG. 6. Theoretical and observed asymptotic transitional distributions $\lim_{n\to\infty} r(x_{n+1} \mid a \leq x_n \leq b, Y_{0,n})$ for Group L. The interval (a, b) for each of the above histograms is indicated by the darkened section on the abscissa of the histogram. The value of N indicated in each histogram indicates the number of cases on which that histogram is based.

The theoretical curves for the one-element model fitted to the histograms in Figs. 5 and 6 were determined by evaluating

$$\lim_{n\to\infty} r(x_{n+1} \mid a \leqslant x_n \leqslant b,\ Y_{0,n}) = \frac{\int_{-\pi}^{\pi} \int_a^b \pi_0(x_n) k(x_{n+1}, z_n) k(x_n, z_n) H(z_n) r(z_n)\, dx_n\, dz_n}{\int_a^b \pi_0(x_n) r(x_n)\, dx_n} \tag{11}$$

with the same numerical methods described previously. For convenience, the function in (11) will be referred to as the transitional density function. Equation 11, it will be noted, contains no new parameters to be estimated. The theoretical prediction, therefore, indicated in Figs. 5 and 6 are parameter free. The darkened interval on the horizontal axis in these figures gives the (a, b) interval within which x_n lies for that histogram.

For the Group S data in Fig. 5, it appears that the fit is reasonably good to the fourth and fifth histograms and reasonably poor to the first three. The one-element identity theory seems to be able to account for the transitions which occur from regions near the center of the reinforcement distribution, but not from regions near the extremes of this distribution. The first histogram in Fig. 5, based on transitions from the most extreme interval, appears to be a close approximation to a uniform distribution, while the theoretical distribution is clearly nonuniform. Histograms 2 and 3 in Fig. 5 show some semblance of a unimodal distribution, but the mode is much closer to the center of the reinforcement distribution than the one-element identity theory predicts. This means that transitions from the extremes to the center occur more frequently than the theory predicts.

These general properties of the Group S histograms in Fig. 5 also characterize the Group L histograms in Fig. 6, only here the disparity between the one-element model and data is much more acute. The theoretical distributions for the extreme intervals not only have modes occurring far from the center of the reinforcement distribution, but in addition have much smaller variances than the corresponding Group S variances. The fourth and fifth histograms in Fig. 6 come close to matching the theoretical modes but the variances of these two histograms are much too large. Thus the data summarized in Figs. 5 and 6 highlight the limitations of the one-element identity theory. Since it is quite likely that responses in the upper histograms in Figs. 5 and 6 (the histograms based on extreme intervals) are based on earlier trials, the referee of this paper suggested that the poor fit of the model to these upper histograms may be due to the prevalence of pre-asymptotic responses, or, perhaps more precisely, to responses that are more pre-asymptotic than those in the lower histograms of these figures.

Extending the identity theory to N elements in this case has the effect of shifting the theoretical means in Figs. 5 and 6 closer to the reinforcement distribution and increasing the theoretical variances. Both modifications bring the theory closer in line with the data. Fortunately, as in the case of determinate reinforcement, a simple relation holds between the N-element and one-element conditional densities, namely

$$j_{\infty}^{(N)}(x_{n+1} \mid x_n , Y_{0,n}) = \frac{1}{N} j_{\infty}^{(1)}(x_{n+1} \mid x_n , Y_{0,n}) + \frac{N-1}{N} r(x_{n+1}). \tag{12}$$

In other words, the N-element conditional density is a linear combination of the predictions of the one-element model and the asymptotic response distribution $r(x)$. The rationale of the relationship is apparent. When the same element is sampled on trials n and $n + 1$, then the one-element model may be used to make predictions—and the probability of such a sampling of the same element on two successive trials is just $1/N$. On the other hand, if the element sampled on trial n is not the same as the element sampled on trial $n + 1$, then the actual response x_n on trial n has no direct effect on response x_{n+1} and the asymptotic response distribution $r(x_{n+1})$ is applicable. The derivation of (12) is given in the Appendix. It should be noted that (12) only holds at asymptote.

By using (12) the theoretical predictions of the N-element model are shown in Figs. 5 and 6. For each of the two groups, a least-squares estimate of N was made over all five intervals for x_n, corresponding to the five histograms for each group. It is clear that the fits are in general much better than those of the theoretical curves of the one-element model, even though definite discrepancies still exist. In the case of Group S the least-squares estimate of N is 2.25, which is close to similar estimates obtained in a variety of experiments. For example, the estimated N was 3.13 in the noncontingent bimodal reinforcement condition studied by Suppes, Rouanet, Levine, and Frankmann (1964). The fits also were remarkably close, with a χ^2 of 313.2 in the earlier study, and a χ^2 of 315.6 in the present study, for essentially the same number of degrees of freedom. (A remark on the interpretation of these large χ^2's has already been made above.)

The results for Group L are more unusual. First, as is evident from a comparison of Figs. 5 and 6, the fit of the theoretical curves of the N-element model to the histogram is not as good for Group L as for Group S. This is reflected in the χ^2 test, which has a value of 736.2. Moreover, the estimate of N is 28.18, which is about the largest value obtained in any experiment known to us. The psychological significance of this large value for the estimated number of stimulus elements is obvious. In the case of Group L, the group with the large target, the effect of the last response should be insignificant, because the probability of resampling the same stimulus element again is only .035, or $1/N$.

It is apparent that the N-element identity model is the best of the several proposed in this paper, and yet is by no means fully satisfactory. We have pushed as far as we consider practical, variations on the fundamental ideas set forth here and in our earlier theoretical study (Suppes and Zinnes, 1961). It would seem that substantial improvements in accounting for behavior in the kind of continuous-response task we have considered must depend on a new theoretical approach.

APPENDIX

N-Element Identity Model

The properties of the N-element identity model that are useful in this paper can be summarized in three theorems. We state these theorems here and in somewhat abbreviated form indicate their proofs.

The notation follows that used in the main body of the paper, except for the introduction of some additional subscripts. Now $z_{i,n}$ is the mean of the distribution $k(x, z_{i,n})$ associated with stimulus element i on trial n, and $g_i(z_i)$ is its asymptotic density. Where no confusion results we drop one of the "i" subscripts and write either $g_i(z)$ or $g(z_i)$. Also when it is necessary to distinguish between a one-element and an N-element result we use superscript. Thus $r^{(1)}(x)$ and $r^{(N)}(x)$ are the asymptotic response densities predicted by the one-element and N-element models, respectively.

Theorem 1. *The asymptotic density*

$g_i(z)$ *for the* i^{th} *element is equal to*

$$g_i(z) = \frac{\pi_1(z)r(z)}{\int \pi_1(x)k(x, z)\, dx} \tag{1}$$

and hence

$$g_1(z) = g_2(z) = \cdots = g_N(z).$$

Proof. On the n^{th} trial, if the i^{th} element is not sampled then we assume $z_{i,n+1} = z_{i,n}$. If the i^{th} element is sampled then all of the previously discussed assumptions of the identity model apply. That is, if the reinforcement (a hit) occurs and it is effective, then $z_{i,n+1} = x_n$, where x_n is the response occurring on the n^{th} trial. If the reinforcement is not effective or if no reinforcement occurs then $z_{i,n+1} = z_{i,n}$. Thus letting

$$\begin{aligned}
Y_{1,n} &= \text{reinforcement occurs on } n^{\text{th}} \text{ trial (a hit)}\\
Y_{0,n} &= \text{reinforcement does not occur on } n^{\text{th}} \text{ trial (a miss)}\\
F_{1,n} &= \text{reinforcement is effective}\\
F_{0,n} &= \text{reinforcement is not effective}\\
s^{(i)}_{1,n} &= \text{element } i \text{ is sampled on trial } n\\
s^{(i)}_{0,n} &= \text{element } i \text{ is not sampled on trial } n
\end{aligned}$$

and assuming

$$P(F_{1,n}) = \theta = 1 - P(F_{0,n})$$

$$P(s_{1,n}^{(i)}) = \frac{1}{N} = 1 - P(s_{0,n}^{(\ )}) \tag{2}$$

we have

$$g_{i,n+1}(z) = \sum_{j,k,l=0}^{1} j_{i,n-1}(z, s_{j,n}^{(i)}, F_{k,n}, Y_{l,n}) \tag{3}$$

$$= j_{i,n+1}(z, s_{1,n}^{(i)}, F_{1,n}, Y_{1,n}) + j_{i,n-1}(z, s_{1,n}^{(\ )}, F_{0,n}, Y_{1,n})$$

$$+ j_{i\ n+1}(z, s_{1,n}^{(i)}, Y_{0,n}) + j_{i,n+1}(z, s_{0,n}^{(i)})$$

$$= \frac{\theta}{N}\pi_1(z)\int k(z, z')g_{i,n}(z')\,dz' + \frac{1-\theta}{N} g_{i,n}(z)\int \pi_1(x_n)k(x_n, z)\,dx_n$$

$$+ \frac{1}{N} g_{i,n}(z)\int \pi_0(x_n)k(x_n, z)\,dx_n - \frac{N-1}{N} g_{i,n}(z). \tag{4}$$

At asymptote

$$g_{i,n}(z) = g_{i,n+1}(z) = g_i(z)$$

so that (4) implies

$$g_i(z) = \frac{\pi_1(z)\int k(z, z')g_i(z')\,dz'}{\int \pi_1(x_n)k(x_n, z)\,dx_n}. \tag{5}$$

Equation 5 is an integral equation and from it we may immediately conclude, since i is an arbitrary element, that

$$g_1(z) = g_2(z) = \cdots = g_N(z). \tag{6}$$

Furthermore, since

$$r(x) = \frac{1}{N}\sum_{i=1}^{N}\int k(x, z)g_i(z)\,dz \tag{7}$$

we have for any element i

$$r(x) = \int k(x, z)g_i(z)\,dz. \tag{8}$$

Replacing the variable of integration in (8) by z' and x by z and using (5) we obtain (1), which completes the proof.

The significance of (1) is that it is equivalent to the corresponding one-element result. (See Eq. 47, Suppes and Zinnes, 1961.) Thus the density $g_i(z)$ for the i^{th} element in the N-element model equals the density $g(z)$ for the single-element theory. From (8) it follows therefore that the asymptotic response densities $r^{(1)}(x)$ and $r^{(N)}(x)$ for these two models are equal.

THEOREM 2. *At asymptote the means $z_1, \ldots, z_N$ of the distributions $k(x, z_1), \ldots, k(x, z_N)$ are independently distributed, that is,*

$$j_\infty(z_1, \ldots, z_N) = g(z_1) \ldots g(z_N). \tag{9}$$

PROOF. The initial steps here follow the usual line of argument and may therefore be briefly summarized as follows:

$$\begin{aligned} j_{n+1}(z_1, \ldots, z_N) = & \sum_{i=1}^{N} j_{n+1}(\ldots z_i \ldots, s_{1,n}^{(i)}, F_{0,n}) \\ & + \sum_{i=1}^{N} j_{n+1}(\ldots z_i \ldots, s_{1,n}^{(i)}, F_{1,n}, Y_{1\,n}) \\ & + \sum_{i=1}^{N} j_{n+1}(\ldots z_i \ldots, s_{1,n}^{(i)}, F_{1,n}, Y_{0,n}) \end{aligned} \tag{10}$$

$$\begin{aligned} = & \sum_{i=1}^{N} \left(\frac{1-\theta}{N}\right) j_n(\ldots z_i \ldots) \\ & + \sum_{i=1}^{N} \left(\frac{\theta}{N}\right) \pi_1(z_i) \int k(z_i, z_i') j_n(\ldots z_i' \ldots)\, dz_i' \\ & + \sum_{i=1}^{N} \left(\frac{\theta}{N}\right) j(\ldots z_i \ldots) \int \pi_0(x) k(x, z_i)\, dx. \end{aligned} \tag{11}$$

At asymptote

$$j_{n+1}(z_1, \ldots, z_N) = j_n(z_1, \ldots, z_N) = j_\infty(z_1, \ldots, z_N)$$

so that (11) implies

$$j_\infty(z_1, \ldots, z_N) = \frac{\sum_{i=1}^{N} \pi_1(z_i) \int k(z_i, z_i') j_\infty(\ldots z_i' \ldots)\, dz_i'}{\sum_{i=1}^{N} \int \pi_1(x) k(x, z_i)\, dx}. \tag{12}$$

Equation 12 is an integral equation and its solution if it exists is unique up to a similarity transformation. It is sufficient therefore to show that (9) is a solution of (12). Thus, noting from (8) that

$$r(z_i) = \int k(z_i, z_i') g(z_i)\, dz_i$$

letting

$$h(z_i) = \int \pi_1(x) k(x, z_i)\, dx \tag{13}$$

and assuming (9), we obtain from (12)

$$j_\infty(z_1, ..., z_N) = \frac{\sum_i \pi_1(z_i)g(z_1) ... r(z_i) ... g(z_N)}{\sum_i h(z_i)}. \tag{14}$$

But from (1)

$$g(z_i)h(z_i) = \pi_1(z)r(z) \tag{15}$$

so that we infer from (14)

$$\begin{aligned} j_\infty(z_1, ..., z_N) &= \frac{\sum_i h(z_i)g(z_1) ... g(z_i) ... g(z_N)}{\sum_i h(z_i)} \\ &= \frac{[g(z_1) ... g(z_N)] \sum_i h(z_i)}{\sum_i h(z_i)} \\ &= g(z_1) ... g(z_N), \end{aligned}$$

whence (9) is a solution of (12), as desired.

THEOREM 3. *At asymptote*

$$j_\infty^{(N)}(x_{n+1} \mid x_n, Y_{0,n}) = \frac{1}{N} j_\infty^{(1)}(x_{n+1} \mid x_n, Y_{0,n}) + \left(\frac{N-1}{N}\right) r(x_{n+1}),$$

where $j_\infty^{(1)}(x_{n+1} \mid x_n, Y_{0,n})$*, the asymptotic transitional density function for a one-element identity model, is equal to*

$$\frac{1}{r(x_n)} \int k(x_{n+1}, z)k(x_n, z)g(z)\, dz.$$

PROOF. We consider first the joint density $j(x_{n+1}, x_n, Y_{0,n})$ which we expand as follows.

$$j^{(N)}(x_{n+1}, x_n, Y_{0,n}) = \sum_i \int \cdots \int j(x_{n+1}, s_{1,n+1}^{(i)}, Y_{0,n}, x_n, z_1, ..., z_N)\, dz_1 ... dz_N, \tag{16}$$

where $z_1, ..., z_N$ are the n^{th} trial (not the $n + 1^{st}$ trial) means associated with the N-elements. Conditionalizing the integrand in (16) and using the fact that

$$j(x_n \mid z_1, ..., z_N) = \frac{1}{N} \sum_{i=1}^{N} k(x_n, z_i) \tag{17}$$

and

$$j(x_{n+1} \mid s_{1,n+1}^{(i)}, Y_{0,n}, x_n, z_1, ..., z_N) = k(x_{n+1}, z_i), \tag{18}$$

we have

$$j(x_{n+1}, s^{(\ )}_{1,n+1}, Y_{0,n}, x_n, z_1, ..., z_N)$$

$$= k(x_{n+1}, z_i) \frac{1}{N} \pi_0(x_n) \frac{1}{N} \sum_{j=1}^{N} k(x_n, z_j) j(z_1, ..., z_N). \quad (19)$$

Integrating (19) over z_i and using the asymptotic result

$$j_\infty(z_1, ..., z_N) = g(z_1) \ldots g(z_N),$$

we see that the right-hand side of (19) becomes

$$\frac{1}{N^2} \pi_0(x_n) \int k(x_{n-1}, z_i) k(x_n, z_i) g(z_i)\, dz_i$$

$$+ \frac{1}{N^2} \pi_0(x_n) \int \cdots \int k(x_{n+1}, z_i) \sum_{j \neq i} k(x_n, z_j) \prod_{m=1}^{N} g(z_m)\, dz_m . \quad (20)$$

The second term in (20) equals

$$\frac{1}{N^2} \pi_0(x_n) r(x_{n+1}) \sum_{j \neq i} k(x_n, z_j) g(z_j)\, dz_j = \left(\frac{N-1}{N^2}\right) \pi_0(x_n) r(x_{n+1}) r(x_n). \quad (21)$$

Thus, from (16), (20) and (21), we have

$$j_\infty^{(N)}(x_{n+1}, x_n, Y_{0,n}) = \left(\frac{\pi_0(x_n)}{N}\right) \int k(x_{n+1}, z_i) k(x_n, z_i) g(z_i)\, dz_i$$

$$+ \left(\frac{N-1}{N}\right) \pi_0(x_n) r(x_{n+1}) r(x_n). \quad (22)$$

From (22), it may be seen that when $N = 1$

$$j_\infty^{(1)}(x_{n+1}, x_n, Y_{0,n}) = \pi_0(x_n) \int k(x_{n+1}, z) k(x_n, z) g(z)\, dz \quad (23)$$

so that we may write (22) as

$$j_\infty^{(N)}(x_{n+1}, x_n, Y_{0,n}) = \frac{1}{N} j_\infty^{(1)}(x_{n+1}, x_n, Y_{0,n}) + \frac{N-1}{N} \pi_0(x_n) r(x_{n+1}) r(x_n). \quad (24)$$

For the asymptotic conditional density we divide (24) by $j_\infty(x_n, Y_{0,n})$, which for both one-element and N-element models is equal to $\pi_0(x_n) r(x_n)$, since from (8), $r^{(1)}(x_n) = r^{(N)}(x_n)$. Thus

$$j_\infty^{(N)}(x_{n+1} \mid x_n, Y_{0,n}) = \frac{1}{N} j_\infty^{(1)}(x_{n+1} \mid x_n, Y_{0,n}) + \frac{N-1}{N} r(x_{n+1}),$$

which was to be proved. From (24) it may also be easily seen that

$$j_{\infty}^{(N)}(x_{n+1} \mid a \leqslant x_n \leqslant b, Y_{0,n}) = \frac{1}{N} j_{\infty}^{(1)}(x_{n+1} \mid a \leqslant x_n \leqslant b, Y_{0,n}) + \frac{N-1}{N} r(x_{n+1}) \quad (25)$$

and that

$$j_{\infty}^{(N)}(x_{n+1} \mid Y_{0,n}) = \frac{1}{N} j_{\infty}^{(1)}(x_{n+1} \mid Y_{0,n}) + \frac{N-1}{N} r(x_{n+1}). \quad (26)$$

Finally, the desired expression for $j_{\infty}^{(1)}(x_{n+1} \mid x_n , Y_{0,n})$ follows at once from (23) and the observation just made that $j_{\infty}(x_n , Y_{0,n}) = \pi_0(x_n)r(x_n)$.

It is of some interest to compare (25) and (26) with the results that obtain for determinate, noncontingent reinforcement (see Suppes, *et al.*, 1964). The exact analogue of (25) holds but not that of (26). In the determinate, noncontingent case, when past reinforcements only are considered, the conditional distribution of responses is identical for all N, and for the linear model as well.

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