Reprinted from
Econometrica
Vol. 24, No. 3, July 1956

A FINITISTIC AXIOMATIZATION OF SUBJECTIVE PROBABILITY AND UTILITY[1]

BY DONALD DAVIDSON AND PATRICK SUPPES

The present paper gives an experimentally oriented theory of decision making. It is shown that the axioms lead to the separate measurement of subjective probability and utility.

1. INTRODUCTION

IN EXPERIMENTING with subjects the authors have been led to construct a theory of rational decision making under conditions of risk which will be closer than previous theories to the empirical circumstances of decision making.[2] Von Neumann and Morgenstern's axiomatization of utility uses a quantitative concept of probability which is not given a behavioristic interpretation in terms of empirical operations. The only obvious way of overcoming this difficulty is to identify subjective and objective probability, as was done by Mosteller and Nogee [4]. Since this identification is almost certainly not justified empirically, the problem arises of simultaneously axiomatizing probability and utility. A formal theory dealing with this problem was provided as long ago as 1931 in a largely unremarked paper of F. P. Ramsey [6]. An alternative approach, partly derivative from ideas of de Finetti, has recently been developed by Savage [7]. While these theories may be satisfactory for normative purposes, from an empirical point of view they share the following disadvantage: for verification, and therefore for the derivation of measures, they require an infinite number of choices, yet no one can ever compare an infinite list of alternatives. Justification of such infinite sets in physics (for example, intervals of time, twice differentiable paths, and continuous forces) plausibly rests on the high degree of approximation with which measurements, based on physical theory, can be carried out. In the domain of the theory of rational choice, it is at present a very open question whether there are empirically applicable quantitative concepts. Until there is impressive evidence that such measurement is possible, the idealization represented by the introduction of infinite sets is without a firm foundation.

The model given here follows the general strategy suggested by Ramsey: first find a chance event with subjective probability of one-half, then use this event to determine the utility of outcomes, and finally use the constructed utility function to measure subjective probabilities. The model differs from Ramsey's in that: (1) the basic set of outcomes and their probability combinations is finite, (2) outcomes are equally spaced in utility, and (3) choices are never required between a sure-thing option and a gamble. This last feature offers a partial solution to the experimental problem of eliminating or cancelling out a specific

[1] Research done in connection with this paper was supported in part by the Office of Ordnance Research, U. S. Army, and in part by the Stanford Value Theory Project.

[2] A detailed report of the experimental work related to the formal developments of the present paper will be found in [2].

utility of participation or gambling. It will be clear that anyone who satisfies the Ramsey model will satisfy the model given in this paper; *a fortiori* anyone who satisfies the von Neumann and Morgenstern model satisfies the model given here, but the converses do not follow. The relative weakness of the present theory is the price paid for making it more behavioristic.

Our theory exploits a simple one-person 2×2 game: the subject must choose between two options, each of which is a two-element probability combination, and where all the probabilities (in one play of the game) are created by the same chance event. In matrix representation, the subject chooses a column, while the event E determines the row and x, y, u and v are possible outcomes of a particular play of the game:

(1.1)

	Option 1	Option 2
E	x	u
$\bar{E}$	y	v

Where the subject is indifferent between the options, the assumption that he chooses to maximize expected utility suggests that we represent the situation by the following equation (where ϕ is his utility function and s his subjective probability function):

$$s(E)\phi(x) + s(\bar{E})\phi(y) = s(E)\phi(u) + s(\bar{E})\phi(v). \tag{1.2}$$

Application of our theory requires that some event (call it E^*) be found such that for every pair of alternatives x and y, we have:

$$s(E^*)\phi(x) + s(\bar{E}^*)\phi(y) = s(E^*)\phi(y) + s(\bar{E}^*)\phi(x). \tag{1.3}$$

For $\phi(x) \neq \phi(y)$ we immediately infer from (1.3):

$$s(E^*) = s(\bar{E}^*).$$

(Intutitively, E^* has subjective probability of $\frac{1}{2}$.) Thus when $E = E^*$, we obtain from (1.2):

$$\phi(x) + \phi(y) = \phi(u) + \phi(v), \tag{1.4}$$

and from (1.4) in turn we may derive equality of utility differences.

As the elimination of subjective probabilities from (1.4) suggests, a finite number of plays of the game using the chance event E^* can be made to yield interval measurement of utility for a finite set of basic alternatives, provided certain simple conditions are met. Important among these conditions is one which requires that the difference in utility between any two preference-adjacent alternatives be the same as the difference in utility between any other two preference-adjacent alternatives (see Definition 2.1 and Axiom A9 below).[3]

Once a utility function defined over the basic alternatives has been found using

[3] Axiomatizations of utility which remove this restriction, but at the cost of requiring that K be infinite, are given in [1] and [8].

the special chance event E^*, it is possible to find chance events with certain specified subjective probabilities. We assume that for any chance event E, $s(E) + s(\tilde{E}) = 1$. Then equation (1.2) may be rewritten:

$$s(E) = \frac{\phi(v) - \phi(y)}{\phi(v) - \phi(y) + \phi(x) - \phi(u)} . \tag{1.5}$$

Let us suppose any given individual has a fixed utility for gambling, and that this utility attaches to any option which involves a risk, but not to sure-thing options (i.e., options whose two outcomes are the same). On this assumption, distortion due to the utility of gambling enters only when one option is risky and one is not. In the present theory, distortion of this sort is eliminated by stipulating that no sure-thing options be used. This restriction does not prevent the achievement of interval measurement and only slightly complicates the theory from a formal point of view (see Axiom A4 and proof of Part B of Theorem 5.1). Of course, we do not mean to suggest that the simple assumption made at the beginning of this paragraph is realistic in the sense of fully accounting for all complexities of gambling behavior.

2. PRIMITIVE AND DEFINED NOTIONS

Our axiomatization is based on six primitive notions: (1) a finite set K of alternatives available to a given individual at a given time; (2) a set X of elements, which may be thought of as occurring with some probability; (3) $\mathfrak{F}$, a family of subsets of X (following the usual terminology, the elements of $\mathfrak{F}$ will be called *events*, e.g., the event of obtaining an odd number with one throw of a die); (4) a binary relation P of preference whose field is K; (5) a quinary relation M, which is used to formalize the situation in which the subject is indifferent between the options shown in (1.1) (i.e., $x, y\ M(E)\ u, v$ if and only if the individual in question is indifferent between receiving x if E occurs and y if $\tilde{E}$ occurs, and receiving u if E occurs and v if $\tilde{E}$ occurs); (6) E^*, an element of $\mathfrak{F}$ (as already indicated, the intended interpretation of E^* is that it is an event with subjective probability ½).

It is common for economists to object that indifference between options (as required for the interpretation of the primitive relation M) is not behavioristically definable. It does not seem clear that one can rule out a priori such a definition. In fact Mosteller and Nogee (in [**4**]) use the intuitively plausible idea that a subject is indifferent between two options when he accepts each fifty per cent of the time. A second approach, described in Davidson, Siegel, and Suppes [**2**], relies on a method of systematic approximation which in turn depends on a behavioristic interpretation of preference. It would certainly be closer to the experimental spirit of the present paper if instead of axiomatizing indifference between options we had directly axiomatized the method of approximation. This approach, however, raises difficult formal problems; in fact there are good reasons for thinking that no finite list of axioms can characterize this method.

Two defined notions are needed for the simple statement of the axioms.

DEFINITION 2.1: *xJy if and only if xPy and for every z in K if xPz then $y = z$ or yPz.*

The intuitive interpretation of the relation J is that xJy if and only if y is the unique immediate successor of x with respect to the relation P.

We need, for the next defined notion, the concept of a *power* of J, which is ordinarily defined recursively:

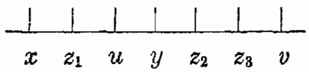
xJ^1y if and only if xJy;

xJ^ny if and only if there is a z such that $xJ^{n-1}z$ and zJy.

The intuitive interpretation of the assertion that xJ^ny is that y is the nth successor of x under the preference relation P.

DEFINITION 2.2: *$r(x, y; u, v) = \alpha$ if and only if there are nonnegative integers m and n such that*: (i) $\alpha = m/n$; (ii) $x \neq y$ *or* $u \neq v$; (iii) *either xJ^my and uJ^nv; or yJ^mx and vJ^nu; or $x = y$, $m = 0$ and (uJ^nv or vJ^nu); or $u = v$, $n = 0$ and* (xJ^my or yJ^mx).

We call r the *ratio function*, and its intuitive interpretation is straightforward: the ratio of the number of intervals between x and y to the number between u and v is α. Thus, if we have alternatives equally spaced in utility as shown:



then $r(x, y; u, v) = 3/4$. Since K is finite the number of elements between any two elements is finite, and the function is defined for all quadruples of elements of K satisfying (ii).[4] The rather complicated condition (iii) on the function is motivated by the desirability of excluding such cases as xPy and vPu. This restriction, which has nothing directly to do with the ratio between (x, y) and (u, v), appreciably simplifies the statement of one of the axioms (Axiom A10 below). The ratio function is not needed in the construction of a numerical utility function, but it or its equivalent is required for the subjective probability function.

3. AXIOMS

We may now state the axioms for what we call finitistic rational choice structures. When we say below that $\mathfrak{F}$ is closed under *complementation*, we mean that if E is in $\mathfrak{F}$, then the complement of E is in $\mathfrak{F}$. $K \times K$ is the *Cartesian product* of K with itself, that is, the set of all ordered couples whose first and second members are in K. The notion easily generalizes to any number of sets as in the case of $K \times K \times \mathfrak{F} \times K \times K$ below. In Axiom A11, "$\subseteq$" is, of course, the symbol for set-theoretical inclusion.

DEFINITION 3.1: *Let K be a finite set, and let X be a set and $\mathfrak{F}$ a family of subsets of X which is closed under complementation and of which X is a member. Let P be a subset of $K \times K$ and let M be a subset of $K \times K \times \mathfrak{F} \times K \times K$. Let E^* be a member of $\mathfrak{F}$. Then* $\langle K, X, \mathfrak{F}, P, M, E^*\rangle$ is a Finitistic Rational Choice

[4] When $n = 0$, $\alpha = \infty$.

Structure *if and only if for every x, y, u, v, x', y', u', and v' in K, and for every E and E' in $\mathfrak{F}$:*

Axiom A1: *P is a simple ordering of K;*

Axiom A2: *If $x \neq y$, then $x, y\ M(E^*)\ y, x$;*

Axiom A3: *If $x, y\ M(X)\ u, v$, then $x = u$;*

Axiom A4: *If $x, y\ M(E)\ u, v$, then $x \neq y$ and $u \neq v$;*

Axiom A5: *If $x, y\ M(E)\ u, v$, then $u, v\ M(E)\ x, y$;*

Axiom A6: *If $x, y\ M(E)\ u, v$, then $y, x\ M(\bar{E})\ v, u$;*

Axiom A7: *If $x, y\ M(E)\ u, v$ and $u, v\ M(E)\ u', v'$, then $x, y\ M(E)\ u', v'$;*

Axiom A8: *If $x, y\ M(E)\ u, v$ and $y \neq v$ and $x\ P\ u$, then $v\ P\ y$;*

Axiom A9: *If $x\ J\ y$, $u\ J\ v$, $x \neq v$ and $u \neq y$ then, $x, v\ M(E^*)\ u, y$;*

Axiom A10: *If $r(x, y; u, v) = r(x', y'; u', v')$ and $x, v\ M(E)\ y, u$ and $x \neq v'$ and $y' \neq u'$, then $x', v'\ M(E)\ y', u'$;*

Axiom A11: *If $x, y\ M(E)\ u, v$ and $x', y'\ M(E')\ u', v'$ and $E \subseteq E'$, then $r(v, y; x, u) \leq r(v', y'; x', u')$, provided $x \neq u$ or $y \neq v$, and $x' \neq u'$ or $y' \neq v'$;*

Axiom A12: *There are elements x, y, u and v in K such that $x, y\ M(E)\ u, v$ and, $x \neq u$ or $y \neq v$.*

The first axiom simply asserts that the relation P of preference is asymmetric, transitive and connected in K. Some economists have objected that connectedness of P excludes the possibility of indifference or equivalence of preference between two distinct alternatives. But it is a mistake to suppose that the formal assumption of the connectedness of P prohibits interpretations which admit indifference. If we begin with a set K which has elements for which the individual in question has equal preference, then we may construct a new set K' whose elements are equivalence classes of elements of K—two elements of K being in the same equivalence class if and only if they are equal in preference. Our axioms then apply to the set K'. If this construction is accepted, no confusion of interpretation should result.[5]

[5] The interpretation of the basic alternatives as equivalence classes was explicitly suggested by Ramsey, for whom "values" are classes of equally preferable possible worlds ([6], p. 178), and more or less implicitly assumed by von Neumann and Morgenstern ([5], see especially p. 617). The point has been the subject of considerable discussion in the economic literature (see, for example, several articles in *Econometrica*, Vol. 20, No. 4 (Oct., 1952)); the discussion has, in our opinion, been partly confused by failure to distinguish between the question whether the assumptions necessary to justify the treatment of the

A few comments on the axioms will perhaps be helpful. Axiom A2 yields the desired result that the subjective probability of E^* is $\frac{1}{2}$. The condition that x and y be distinct is needed to avoid contradiction with A4, the axiom which excludes the consideration of sure-thing gambles. Since the chance event X must occur with certainty, the meaning of A3 is clear; only x and u matter in determining the equivalence of preference of (x, y) and (u, v). Axioms A5 and A7 assert the symmetry and transitivity of equivalent options, and A6 expresses an expected principle of symmetry between E and its complement. Axiom A8 says that if the option (x, y) is equivalent to the option (u, v) and x is preferred to u then v is preferred to y. The condition that y and v be distinct is inserted to take care of the limiting case when the subjective probability of E is zero. Axiom A9′ is the axiom from which we get the equal spacing in utility of elements of K. It says essentially the following: if y is the immediate successor of x, and v is the immediate successor of u, then the utility difference between x and y is equivalent to the utility difference between u and v. The conditions that x and v be distinct and that u and y be distinct are needed to avoid contradiction with A4, as in the case of A2. The function of Axiom A10 is to guarantee that a person satisfying the axioms acts as if he had a subjective probability for any chance event E independent of what particular stakes are connected with the occurrence of E or its complement. This axiom corresponds closely to some informal requirements of Ramsey (see [**6**], p. 179). The eleventh axiom imposes in effect the reasonable requirement that a person will bet at least as much on event E' as E if whenever E occurs, E' must also occur. Since the ratio function is used only in A10 and A11, it will perhaps clarify its intuitive significance by considering a simple example which brings out its role in A11. On the basis of A1-A9 (in fact, just A1, A2, A4, A5, A7, A8 and A9) we can prove the existence of a numerical utility function ϕ on K which is unique up to a linear transformation. For purposes of illustration, let $\phi(x) = 4$, $\phi(y) = \phi(y') = 8$, $\phi(v) = \phi(v') = 12$, and $\phi(u) = \phi(u') = -8$, and suppose that $x, y\ M(E)\ u, v$ and $x', y'\ M(E')\ u', v'$ and $E \subseteq E'$. Now ϕ has the property that

$$r(v, y; x, u) = \frac{\phi(v) - \phi(y)}{\phi(x) - \phi(u)} = \frac{12 - 8}{4 - (-8)} = \tfrac{1}{3}.$$

Since $E \subseteq E'$, a person would be obviously foolish concerning his evaluation of the probability of E and E' if he did not choose x' so that $\phi(x') \leqslant \phi(x)$. But if $\phi(x') \leqslant \phi(x)$, then $r(v, y; x, u) \leqslant r(v', y'; x', u')$, the reasonableness of which we wanted to argue.[6] The last axiom (A12) simply guarantees there are elements

indifference relation as a congruence relation are justified at all, and the relatively trivial question whether these assumptions are formally included in the axioms or preformally assumed in the construction of equivalence classes. Samuelson's "strong independence" axiom expresses the kind of property indifference must have in order to be a congruence relation.

[6] This discussion assumes the subject knows or believes that $E \subseteq E'$. Under conditions where this assumption is questionable, it is possible to weaken A11 by rewriting "$\subseteq$" and "$\leqslant$" as equalities; the only consequence will be that it will be impossible to prove that the subjective probability function has the monotonicity property expressed in Theorem 5.1, Part (A-v) below.

in K which permit us to determine the subjective probability of every element in $\mathfrak{F}$. Since the structure of K is completely determined by the number of elements in K, this axiom amounts to a restriction on the kind of chance events in $\mathfrak{F}$. The requirement that $x \neq u$ or $y \neq v$ assures that the axiom is not merely guaranteeing the trivial case $x, y\ M(E)\ x, y$, which holds for any E and any two distinct elements x and y.

Some readers may be interested in the relation between our axiomatization and the notions introduced in Savage's recent book ([7]). The set X in our approach corresponds to the set of states of nature, and the set K to the set of consequences. For Savage, a gamble is a decision, mapping X into K, which is constant with respect to some partition of X. Our options are even more special decisions, namely gambles with respect to some two-element partition of X. Any element of $\mathfrak{F}$ and its complement constitutes such a partition. The M relation corresponds to indifference between options. Contrary to Savage's approach, we do not assume a simple ordering on options, but may derive it as a consequence. Axiom A1 may be interpreted as requiring that the options whose consequences are independent of the state of nature be simply ordered. It is obvious, of course, that the special character of our axiomatization is what permits this weakening of the ordering axioms.

4. ELEMENTARY THEOREMS

The elementary theorems listed in this section are useful in shortening the proof of the adequacy of the axioms in the next section. Since most of these theorems are easy consequences of the axioms, the proofs of all but the last are omitted.

Throughout this section the statement of the condition that elements x, y, u, v and the like be in K is omitted for brevity.

THEOREM 4.1: *If $x, y\ M(E^*)\ u, v$ and $u\ P\ x$, then $y\ P\ v$.*

THEOREM 4.2: *If $x, y\ M(E^*)\ x, z$, then $y = z$.*

THEOREM 4.3: *If $x\ J^n y$, then $x\ P\ y$.*

THEOREM 4.4: *If $x\ P\ y$, then there is an n such that $x\ J^n y$.*

THEOREM 4.5: *If $x\ J^n\ y$ and $x\ J^n\ z$, then $y = z$.*

THEOREM 4.6: *If $x\ J^n z$ and $y\ J^n\ z$, then $x = y$.*

THEOREM 4.7: *If $x\ J^m y$ and $y\ J^n z$, then $x\ J^{m+n} z$.*

THEOREM 4.8: *If $x\ J^m y$ and $x\ J^{m+k}\ z$, then $y\ J^k z$.*

THEOREM 4.9: *If $x\ J^{m+k}\ z$ and $y\ J^k z$, then $x\ J^m\ y$.*

THEOREM 4.10: *If $x\ J^{m+k}\ y$, then there is a z such that $x\ J^m\ z$ and $z\ J^k\ y$.*

The slight complication in the proof of the next theorem is due mainly to the elimination of sure-thing bets (Axiom A4).

THEOREM 4.11: *If* $x\ J^n\ y$ *and* $u\ J^n\ v$, *and if* $x \neq v$ *and* $u \neq y$, *then* $x, v\ M(E^*)\ u, y$.

PROOF: For $n = 1$, the proof is immediate from A9. Assume the theorem holds for n. By hypothesis for $n + 1$ there are elements z and w such that $x\ J^n\ z$, $w\ J^n\ v$, $z\ J\ y$ and $u\ J\ w$. *Case* 1. $w = z$. Hence $z\ J^n\ v$. If $n = 1$, then $y = v$ and $x = u$, and hence by hypothesis, by A2, and by A7, it follows that $x, y\ M(E^*)\ u, v$. If $n \neq 1$, then by 4.8 and 4.9, $y\ J^{n-1}\ v$ and $x\ J^{n-1}\ u$, and by the inductive hypothesis $x, v\ M(E^*)\ y, u$. Since $y, u\ M(E^*)\ u, y$, we obtain the desired result by A7. *Case* 2. $w \neq z$. From the inductive hypothesis, $x, v\ M(E^*)\ z, w$; and by A9, $z, w\ M(E^*)\ u, y$. Hence by A7 we obtain the desired result.

Since most of the above elementary theorems are intuitively obvious, we do not always make explicit reference to them in the proof of the main theorem in the next section.

5. ADEQUACY THEOREM

We are now in a position to state and prove the theorem which shows that the axioms for finitistic rational choice structures are adequate to yield a numerical utility function ϕ on K which is unique up to a linear transformation, and a non-additive, numerical subjective probability function s on $\mathfrak{F}$ which is absolutely unique. Furthermore, as Part (*A-vi*) of the theorem below shows, we can summarize a person's behavior when he satisfies the axioms by saying that he is indifferent between the option of x if E occurs and y if $\tilde{E}$, and the option of u if E and v if $\tilde{E}$, if and only if the expected value of ϕ with respect to s is the same in both cases.

Some further remarks on the subjective probability function follow the proof of the theorem. We note here that Part (*A-vi*) of the theorem reflects the prohibition of sure-thing options. The reason for the restriction in Part (B) that the number of elements in K be at least five is that when the number of elements is smaller, there are not sufficient cross-checks to guarantee interval-scale measurement of utility. The proof of (B) brings this point out in detail.

THEOREM 5.1: *Let* $\langle K, X, \mathfrak{F}, P, M, E^* \rangle$ *be a finitistic rational choice structure, and let* n^* *be the number of elements in* K. *Then*:

(A) *There exists an ordered pair* $\langle \phi, s \rangle$ *of real-valued functions with* ϕ *defined on* K *and* s *defined on* $\mathfrak{F}$ *such that for every* x, y, u *and* v *in* K *and every* E *and* F *in* $\mathfrak{F}$:

(i) $x\ P\ y$ *if and only if* $\phi(x) > \phi(y)$,

(ii) $s(E) \geqslant 0$,

(iii) $s(X) = 1$,

(iv) $s(E) + s(\tilde{E}) = 1$,

(v) *If* $E \subseteq F$, *then* $s(E) \leqslant s(F)$,

(vi) $x, y\ M(E)\ u, v$ *if and only if* $\phi(x) \neq \phi(y)$ *and* $\phi(u) \neq \phi(v)$ *and*

$$s(E)\ \phi(x) + s(\tilde{E})\ \phi(y) = s(E)\ \phi(u) + s(\tilde{E})\ \phi(v);$$

(B) *If* $n^* \geqslant 5$ *and if* $\langle \phi_1, s_1 \rangle$ *and* $\langle \phi_2, s_2 \rangle$ *are any two ordered pairs of function satisfying* (*A*), *then:*

(i) $s_1 = s_2$,

(ii) *There exist real numbers* a *and* b *with* $a > 0$ *such that for every* x *in* K

$$\phi_1(x) = a\phi_2(x) + b.$$

PROOF:

Part (A). Let z^* be the first element of K under the ordering P, and let k be an arbitrary positive integer. Since according to (i) we want z^* to be assigned the highest number of any element in K, we define the function ϕ as follows (for every x in K):

$$\phi(z^*) = k,$$

$$\phi(x) = k - n \quad \text{if and only if} \quad z^* \ J^n \, x.$$

We begin by proving (i). Assume, first: $x \ P \ y$. By 4.5 there is an n such that $x \ J^n \ y$. If $x = z^*$ then $\phi(x) = k$ and $\phi(y) = k - n$. If $x \neq z^*$, then there is an m such that $z^* \ J^m \ x$, and thus by 4.8

$$\phi(x) = k - m > k - m - n = \phi(y).$$

Next assume: $\phi(x) = n > m = \phi(y)$. If $n = k$, then $x \ J^{k-m} \ y$ and hence $x \ P \ y$ by 4.4. If $n < k$ then $z^* \ J^{k-n} \ x$ and $z^* \ J^{k-m} \ y$, whence by 4.9 $x \ J^{n-m} \ y$, and thus $x \ P \ y$.

We now use ϕ to define the subjective probability function s: $s(E) = \alpha$ if and only if there are elements x, y, u and v in K such that:

(i) $x, y \ M(E) \ u, v$,

(ii) $\dfrac{\phi(v) - \phi(y)}{\phi(v) - \phi(y) + \phi(x) - \phi(u)} = \alpha$,

(iii) $x \neq u$ or $y \neq v$.

Axiom A12 guarantees that s assigns a probability to every element in $\mathfrak{F}$. The special case of A11 when $E = E'$ assures us that s is a well-defined function. That $s(E) \geqslant 0$ follows easily from A8 and (i); and that $s(X) = 1$ is an immediate consequence of A3. Moreover, (iv) follows from A6. From the definition of ϕ, 2.1 and 2.2, and the elementary theorems on powers of J it is clear that

$$r(x, y; u, v) = \frac{\phi(x) - \phi(y)}{\phi(u) - \phi(v)}, \tag{1}$$

and thus it is easy to show that (v) follows from A11.

If we assume that $x, y \ M(E) \ u, v$, then from A11, (1) and A8 we infer:

$$s(E) = \frac{\phi(v) - \phi(y)}{\phi(v) - \phi(y) + \phi(x) - \phi(u)}$$

and it is then a matter of simple algebra to verify that

$$s(E)\phi(x) + s(\tilde{E})\phi(y) = s(E)\phi(u) + s(\tilde{E})\phi(v). \tag{2}$$

Assuming now (2), we prove the other half of (vi). By Axiom A12 there are elements x', y', u' and v' in K such that

(3) $$x', y' \; M(E) \; u', v'$$

and

$$s(E) = \frac{\phi(v') - \phi(y')}{\phi(v') - \phi(y') + \phi(x') - \phi(u')}.$$

Hence by (2) and (iv)

$$\frac{\phi(x) - \phi(u)}{\phi(v) - \phi(y)} = \frac{\phi(x') - \phi(u')}{\phi(v') - \phi(y')},$$

and thus by (1)

(4) $$r(x, u; v, y) = r(x', u'; v', y').$$

By hypothesis, $\phi(x) \neq \phi(y)$ and $\phi(u) \neq \phi(v)$, and therefore by (3), (4) and A10, $x, y \; M(E) \; u, v$, which concludes the proof of Part (A).

Part (B). Let $\langle \phi_1, s_1 \rangle$ and $\langle \phi_2, s_2 \rangle$ be any two ordered pairs of functions satisfying (A). We first show that ϕ_1 and ϕ_2 are related by a linear transformation.

Let z^{**} be the second element of K under the ordering P. We define two functions h_1 and h_2 (for every x in K):

$$h_1(x) = \frac{\phi_1(x) - \phi_1(z^*)}{\phi_1(z^*) - \phi_1(z^{**})},$$

$$h_2(x) = \frac{\phi_2(x) - \phi_2(z^*)}{\phi_2(z^*) - \phi_2(z^{**})}.$$

We see at once that

(1) $$\begin{cases} h_1(z^*) = h_2(z^*) = 0, \\ h_1(z^{**}) = h_2(z^{**}) = -1, \end{cases}$$

and

(2) $$\langle h_1, s_1 \rangle \quad \text{and} \quad \langle h_2, s_2 \rangle \quad \text{satisfy (A).}$$

Also clearly h_1 is a linear transformation of ϕ_1 and h_2 is a linear transformation of ϕ_2. Consequently we want to show that $h_1 = h_2$. First we notice that by A2 if $x \neq y$ then $x, y \; M(E^*) \; y, x$; therefore we must have

(3) $$s_1(E^*) = s_2(E^*) = \tfrac{1}{2},$$

(for s_1 and s_2 both satisfy (iv) and (vi) of (A)).

Since K is finite, let N be the number of elements in K, and for $n \leqslant N$, let x_n be the nth element of K under the ordering P. Thus $x_1 = z^*$, and $x_2 = z^{**}$. We prove by induction that if $n \leqslant N$ then $h_1(x_n) = h_2(x_n) = -n + 1$. For $n = 1$, we already have (1). Assume this holds for n. The case of $n \geqslant N$ is vacuous, so we suppose $n < N$. The troublesome case is $n = 2$, which we now consider. (The trouble is due to A4; we don't have: $x_1, x_3\, M(E^*)\, x_2, x_2$.) Since by hypothesis $N \geqslant 5$, from 4.11 we obtain: $x_1, x_4\, M(E^*)\, x_2, x_3$; $x_2, x_5\, M(E^*)\, x_3, x_4$; $x_1, x_5\, M(E^*)\, x_2, x_4$; and thus by (3) and (vi) of (A) we have six equations (for $i = 1, 2$):

$$\text{(4)} \qquad \begin{cases} h_i(x_1) + h_i(x_4) = h_i(x_2) + h_i(x_3), \\ h_i(x_2) + h_i(x_5) = h_i(x_3) + h_i(x_4), \\ h_i(x_1) + h_i(x_5) = h_i(x_2) + h_i(x_4). \end{cases}$$

From (1) and (4) we get: $h_1(x_3) = h_2(x_3) = -2$.

The argument for $n > 2$ is straightforward. We have: $x_1\, J\, x_2$ and $x_n\, J\, x_{n+1}$, and since $x_2 \neq x_n$, using A9 we infer that $x_1, x_{n+1}\, M(E^*)\, x_2, x_n$. Hence for $i = 1, 2$,

$$h_i(x_1) + h_i(x_{n+1}) = h_i(x_2) + h_i(x_n),$$

and using our inductive hypothesis we then have:

$$0 + h_i(x_{n+1}) = -1 + (-n + 1).$$

That is, $h_i(x_{n+1}) = -n$, which completes the proof that $h_1 = h_2$.

Let E be an arbitrary element of $\mathfrak{F}$. By A12 and definition of s there are elements x, y, u and v in K such that $\phi_i(x) \neq \phi_i(y)$, $\phi_i(u) \neq \phi_i(v)$, and

$$\text{(5)} \qquad s_i(E) = \frac{\phi_i(v) - \phi_i(y)}{\phi_i(v) - \phi_i(y) + \phi_i(x) - \phi_i(u)}, \qquad \text{for} \quad i = 1, 2.$$

Since we have shown that ϕ_1 and ϕ_2 are related by a linear transformation, it is easily seen by using (5) that $s_1(E) = s_2(E)$, and thus $s_1 = s_2$, which completes the proof of (B), and of the theorem.

We conclude this paper with one or two brief remarks on subjective probability. In the first place, it is clear from A12 and the requirement that K be a finite set that the number of elements in $\mathfrak{F}$ with distinct subjective probabilities is finite. This fact suggests the question: if K has N elements, what numerical subjective probabilities are attainable? It is not difficult to show that the answer is: all probabilities α for which there exist nonnegative integers m and n such that $\alpha = m/(m + n)$ and m and n are less than N. For N of any appreciable magnitude there is a very large number of such probabilities; if β is any real number between 0 and 1, then for K with cardinality N and for a sufficiently ample $\mathfrak{F}$ we may find a subjective probability α such that $|\beta - \alpha| \leqslant 1/(N + 1)$.

It should be obvious why additivity of probability fails in the finitistic set-up analyzed in this paper. In any finitistic rational choice structure there are always sums of probabilities which are not attainable. For example, if $N = 4$ and $\mathfrak{F}$ is ample, then there are events E and F in $\mathfrak{F}$ such that the intersection of

E and F is empty, $s(E) = 1/4$ and $s(F) = 2/5$, but there can be no event E' in $\mathfrak{F}$ with $s(E') = 13/20$.[7]

Stanford University

REFERENCES

[1] DAVIDSON, D., J. C. C. MCKINSEY, AND P. SUPPES: "Outlines of a Formal Theory of Value, I," *Philosophy of Science*, Vol. 22, 1955, pp. 140–160.

[2] DAVIDSON, D., S. SIEGEL, AND P. SUPPES: "Some Experiments and Related Theory on the Measurement of Utility and Subjective Probability," Report No. 4, Stanford Value Theory Project, August 15, 1955.

[3] EDWARDS, W.: "The Theory of Decision Making," *Psychological Bulletin*, Vol. 51, 1954, pp. 380–417.

[4] MOSTELLER, F., AND P. NOGEE: "An Experimental Measurement of Utility," *Journal of Political Economy*, Vol. 59, 1951, pp. 371–404.

[5] VON NEUMANN, J., AND O. MORGENSTERN: *Theory of Games and Economic Behavior*, 2nd ed., Princeton, 1947.

[6] RAMSEY, F. P.: *The Foundations of Mathematics and other Logical Essays*, London, 1931.

[7] SAVAGE, L. J.: *Foundations of Statistics*, New York, 1954.

[8] SUPPES, P., AND M. WINET: "An Axiomatization of Utility Based on the Notion of Utility Differences," *Management Science*, Vol. 1, 1955, pp. 259–270.

[7] The axiomatization given in this paper shows there is no fundamental logical or conceptual problem in the construction of a subjective probability function which is in general non-additive, contrary to the view sometimes expressed (see, e.g., [3]).