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A Linear Model for a Continuum of Responses[1]

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The aim of the present investigation is to formulate and analyze a linear model for simple learning with a continuum of responses. The analogous model for a finite number of responses has been extensively studied both experimentally and mathematically (for references see Chapter 8).

The experimental situation consists of a sequence of trials. On each trial the subject of the experiment makes a response, which is followed by a reinforcing event indicating the correct response for that trial. For the theory we are considering, the relevant data on each trial may be represented by an ordered pair (x, y) of numbers, where x represents the response and y the reinforcement. In the finite case of two responses and two reinforcements, we have simply that x and y are either 1 or 2.) (The phrases "finite case" and "finite model" refer to the linear model for a finite number of responses, and the phrases "continuous case" and "continuous model" to the linear model for a continuum of responses.) The psychological basis of the finite and continuous models is the same, namely, the fundamental postulate that in a simple learning situation with the same stimulus presentation on each trial, systematic changes in response probability are determined by the conditions of reinforcement.

There have been few if any learning experiments of a quantitative character involving a continuum of both responses and reinforcements on each trial, apart from some early line-drawing experiments by Thorndike and recent tracking experiments. However, once the question of performing such experiments is raised, it is very easy to think of a variety of specific experimental situations to test the model to be described here. In order to give the developments to follow a certain concreteness, I shall describe at this point, and later consider in some numerical detail, the apparatus now being used at Stanford University to study the empirical validity of this continuous model. The subject is seated facing a circular disk, which is approximately 24 inches in diameter. He is told that his task on each trial is to predict by means of a pointer where a spot of light will appear on the rim of the disk.

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At the end of each trial the "correct" position of the spot is shown to the subject; this is the reinforcing event for that trial. The objective of the model considered here is the same as that of the finite models already much tested, namely, to predict for each trial, and asymptotically, the probability distribution of responses. (The subject's responses are his pointer predictions.) In the case of the disk, the response number x and the reinforcement number y vary continuously from 0 to 2π.

From the character of this example of a typical experiment to which the model applies, it should be obvious that the model is also relevant to a large number of skill-performance tasks where the independent and dependent variables are both continuous. On the other hand, as the model is formulated in this paper it is not meant to apply directly to measures of response amplitude, where responses and reinforcements are themselves discrete.

For either the finite or the continuous case an experiment may be represented by a sequence $(A_1, E_1, A_2, E_2, \cdots, A_n, E_n, \cdots)$ of random variables, where the choice of letters follows conventions established in the literature: the value of the random variable A_n is the number representing the actual response on trial n, and the value of E_n is the number representing the reinforcing event which occurred on trial n. Any sequence of values of these random variables represents a possible experimental outcome. Hereafter, we write down only finite sequences of these random variables, and it is convenient to write them in reverse order: $(E_n, A_n, E_{n-1}, A_{n-1}, \cdots, E_1, A_1)$.

For both the finite and continuous models the theory is formulated for the probability of a response on trial $n+1$ given the entire preceding sequence of responses and reinforcements. For this preceding sequence we use the notation s_n.[2] Thus, more formally, s_n is a finite sequence of length $2n$ of possible values of the sequence of random variables $(E_n, A_n, E_{n-1}, A_{n-1}, \cdots, E_1, A_1)$. The (cumulative) joint distribution of the first n responses and n reinforcements is denoted by J_n; that is, if $s_n = (y_n, x_n, \cdots, y_1, x_1)$, then

$$J_n(s_n) = J_n(y_n, x_n, \cdots, y_1\, x_1) = P(E_n \leqq y_n, A_n \leqq x_n, \cdots, E_1 \leqq y_1\, A_1 \leqq x_1)\,,$$

where P is the measure on the underlying sample space. For notational clarity we use variables $x_1, x_2, \cdots, x_n, \cdots$ for values of the response random variables and $y_1, y_2, \cdots, y_n \cdots$ for the reinforcement random variables.

Before presenting the continuous model, we shall briefly restate the discrete choice model in random variable notation to facilitate later comparison.

Restricting ourselves in the finite case to the values 1 and 2 for the random variables, we define

$$p_{1,n} = P(A_n = 1)\,,$$
$$p_{s,1,n} = P(A_n = 1 \mid s_{n-1})\,.$$

The linear model for the two-response case is embodied in the following two axioms (a more general formulation requiring more axioms is possible—see Chapter 8, Sec. 3—but will not be considered here).

Axiom F1. *If* $E_n = 1$, *then* $p_{s,1,n+1} = (1-\theta)p_{s,1,n} + \theta$.

[2] The notation s_n corresponds to x_n in Chapter 8.

Axiom F2. *If* $E_n = 2$, *then* $p_{s,1,n+1} = (1-\theta)p_{s,1,n}$.

Here θ is intuitively to be thought of as the learning parameter; formally θ is some real number in the open interval (0, 1).

There are exactly three results for this finite model that we want briefly to mention. The first is that we may derive for the "mean" probabilities $p_{1,n}$ the following recurrence relation (cf. Chapter 8, Equation 5.2):

$$(1) \qquad p_{1,n+1} = (1-\theta)p_{1,n} + \theta P(E_n = 1)\,.$$

A variety of testable experimental results follow from specifying $P(E_n = 1)$. The case most studied is the so-called *noncontingent* one, in which we set $P(E_n = 1) = \pi$, where π is in the interval (0, 1) and is independent of n. For this case Equation 1 has the explicit solution

$$(2) \qquad p_{1,n} = \pi - (\pi - p_{1,1})(1-\theta)^{n-1}$$

and the asymptotic result

$$(3) \qquad p_{1,\infty} = \pi\,,$$

that is, the asymptotic probability of a response is equal to the noncontingent probability of a reinforcement.

The third result deals with the *simple contingent* case, specified by the following conditional probabilities of reinforcement:

$$(4) \qquad P(E_n = 1 \mid A_n = 1) = \pi_1, \qquad P(E_n = 1 \mid A_n = 2) = \pi_2\,.$$

Again the asymptotic result for this special case of Equation 1 is easily established:

$$(5) \qquad p_{1,\infty} = \frac{\pi_2}{1 - \pi_1 + \pi_2}\,.$$

A number of additional statistically testable dependency relations and sequential effects for both the noncontingent case and the contingent case may be derived from Axioms F1 and F2 (cf. Chapter 8, Sec. 8), but the analogues of Equations 1, 2, 3, and 5 will be initially sufficient for our present purposes.

We shall need first more notation for various distributions. It is assumed that the values of all the random variables lie in some interval $[a, b]$ (with possibly $a = -\infty$ and $b = \infty$). In addition to the joint distribution J_n, we denote the marginal distribution of A_n by $R_n(x) = P(A_n \leq x)$, and the marginal distribution of E_n by $F_n(y) = P(E_n \leq y)$.

We shall also use F for certain conditional distributions of E_n. Densities (which as a consequence of our axioms will always exist) are denoted by the corresponding lower-case letters.

Axioms

In the continuous case we have a learning parameter θ, which serves much the same function as the corresponding parameter of the finite model. However, it does not seem reasonable for the full effect of reinforcement to be concentrated at a point as Axiom F1 asserts it is for the finite case.

Consequently we assume in addition to θ a *smearing distribution* $K(x; y)$ which spreads the effect of reinforcement around the point of reinforcement. For each reinforcement y, $K(x; y)$ is a distribution on responses; that is, $K(a; y)=0$ and $K(b; y)=1$, and if $x \leq x'$, then $K(x; y) \leq K(x'; y)$ for every y in $[a, b]$. The reasonable assumptions that the density $k(x; y)$ is unimodal (at y) and that it is symmetric will be investigated in the sequel but not assumed as fundamental axioms. Prior at least to extensive experimentation, these assumptions seem most plausible empirically.

The first two axioms below are simply directed at making explicit assumptions of smoothing properties which seem highly justified empirically. Mathematically weaker assumptions could be made, but this is not a matter of importance here. The third axiom asserts the analogue of Axioms F1 and F2 for the finite case.

Axiom C1. *The distribution* J_n *is continuous and piecewise twice differentiable in each variable.*

Axiom C2. *The distribution* $K(x; y)$ *is continuous and piecewise twice differentiable in both variables.*

Axiom C3. $J_{n+1}(x \mid y_n, x_n, s_{n-1}) = (1-\theta)J_n(x \mid s_{n-1}) + \theta K(x; y_n)$.

Axiom C3 says that given the sequence (y_n, x_n, s_{n-1}), the conditional distribution of the response random variable on trial $n+1$ is $(1-\theta)$ times the conditional distribution of the response random variable on trial n given the sequence s_{n-1}, plus θ times the smearing distribution $K(x; y_n)$. The parameter of the smearing distribution is, of course, the point of reinforcement y_n in the sequence (y_n, x_n, s_{n-1}).

Derivation of Integral Equations for Asymptotic Response Distributions

In this section we want to derive continuous analogues of Equations 1, 2, 3, and 5.

As the analogue of Equation 1, we have the following theorem.

THEOREM.

$$R_{n+1}(x) = (1-\theta)R_n(x) + \theta\int_a^x\int_a^b k(x; y)f_n(y)dx\,dy\,, \tag{6}$$

and if $a \leq a_1 < a_2 \leq b$, *then*

$$\int_{a_1}^{a_2} r_{n+1}(x)dx = (1-\theta)\int_{a_1}^{a_2} r_n(x)dx + \theta\int_{a_1}^{a_2}\int_a^b k(x; y)f_n(y)dx\,dy\,. \tag{7}$$

PROOF.

$$R_{n+1}(x) = \int_a^x r_{n+1}(x)dx$$
$$= \underbrace{\int_a^x\int_a^b \cdots \int_a^x}_{2n\ \text{times}} j_{n+1}(x, s_n)dx ds_n$$

by the definition of the marginal distribution R_n

$$= \int_a^x \int_a^b \cdots \int_a^b j_{n+1}(x \mid y_n, x_n, s_{n-1}) j_n(y_n, x_n, s_{n-1}) dx\, dy_n\, dx_n\, ds_{n-1}$$

by the usual conditional distribution relations

$$= \int_a^x \int_a^b \cdots \int_a^b [(1-\theta) j_n(x \mid s_{n-1}) + \theta k(x; y_n)] \cdot j_n(y_n, x_n, s_{n-1}) dx\, dy_n\, dx_n ds_{n-1}$$

by applying Axiom C3 in terms of densities

$$= (1-\theta) R_n(x) + \theta \int_a^x \int_a^b k(x; y_n) f_n(y_n) dx dy_n \qquad \text{by integrating out.}$$

And Equation 7 follows directly from Equation 6. Q.E.D.

From Equation 6 we easily obtain the continuous analogues of Equations 2 and 3. Here $R(x)$ is the mean asymptotic probability distribution of responses, and $F(y)$ is the reinforcement distribution, which is independent of n.

THEOREM. *In the noncontingent case*

$$(8) \quad R_n(x) = \int_a^x \int_a^b k(x; y) f(y)\, dx\, dy - \left[\int_a^x \int_a^b k(x; y) f(y)\, dx\, dy - R_1(x) \right] (1-\theta)^{n-1}$$

and

$$(9) \qquad \lim_{n \to \infty} R_n(x) = R(x) = \int_a^x \int_a^b k(x; y) f(y)\, dx\, dy\, .$$

PROOF. Since the noncontingent case is specified by a distribution $F(y)$ independent of n, Equation 6 becomes

$$R_{n+1}(x) = (1-\theta) R_n(x) + \theta \int_a^x \int_a^b k(x; y) f(y) dx\, dy\, .$$

The integral on the right is a function $C(x)$ of x alone. Thus the equation may be written

$$R_{n+1}(x) = (1-\theta) R_n(x) + C(x)\, ,$$

and exactly the argument which yields Equations 2 and 3 in the finite case yields Equations 8 and 9 as consequences of this last equation. Q.E.D.

The integral equation 9 for the asymptotic response distribution calls for some comment. If the smearing distribution $K(x; y)$ is known, then the response distribution $R(x)$ may be obtained by direct integration, since the reinforcement distribution $F(y)$ is selected by the experimenter and is thus completely known. A central difficulty, however, is that the smearing distribution is not directly observable, i.e., there is no observable quantity in the experimental set-up which yields an empirical histogram of this distribution. It is possible to infer the smearing distribution from $R(x)$ and $F(y)$ by using Equation 9. From this standpoint, if we assume that the density $k(x; y)$ is a symmetric function of the difference $x - y$, we may differentiate Equation 9, make a change of variable, and get

$$r(x) = \int_{x-b}^{x-a} k(t)f(x-t)\,dt\,, \tag{10}$$

which is an integral equation of Fredholm type with kernel $f(x-t)$. Techniques for solving this equation by using the empirical histogram of $r(x)$ will not be pursued here, but it may be noted that by approximating $r(x)$ and $f(x-t)$ by means of trigonometric polynomials a good approximation of $k(t)$ may be obtained under relatively mild restrictions on degeneracy.

Because the smearing distribution is postulated to be independent of reinforcement distribution, the solution of $k(t)$ from Equation 10 for one experiment may be used to predict the response distribution in other experiments with different reinforcement distributions.

We now turn to analogues of the simple contingent case specified by Equation 4 for the finite model. For the simplest analogue we choose a point c in (a, b): if response $x \leqq c$ is made, we use distribution f_1 for reinforcements; and if response $x > c$ is made, we use distribution f_2. That is, the *simple contingent case* is specified by a triple (c, f_1, f_2) such that

$$f_n(y) = \int_a^c f_1(y)r_n(x)\,dx + \int_c^b f_2(y)r_n(x)\,dx\,. \tag{11}$$

Since $R_n(a) = 0$ and $R_n(b) = 1$, we may integrate Equation 11 to obtain

$$f_n(y) = R_n(c)f_1(y) + [1 - R_n(c)]f_2(y)\,. \tag{12}$$

We may use Equation 12 to obtain the asymptotic response distribution as follows.

THEOREM. *For the simple contingent case as defined by Equation* 11,

$$R(x) = \int_a^x\int_a^b k(x; y)\{R(c)f_1(y) + [1 - R(c)]f_2(y)\}dx\,dy\,, \tag{13}$$

where

$$R(c) = \frac{\int_a^c\int_a^b k(x; y)f_2(y)dx\,dy}{1 - \int_a^c\int_a^b k(x; y)f_1(y)dx\,dy + \int_a^c\int_a^b k(x; y)f_2(y)dx\,dy}\,. \tag{14}$$

PROOF. Substituting Equation 12 into Equation 6, we get

$$R_{n+1}(x) = (1-\theta)R_n(x) + \theta\int_a^x\int_a^b k(x; y)\{R_n(c)f_1(y) + [1 - R_n(c)]f_2(y)\}dx\,dy\,. \tag{15}$$

By rearranging the second term on the right of Equation 15, and letting $x = c$, we have

$$R_{n+1}(c) = (1-\theta)R_n(c) + \theta[R_n(c)g_1(c) + g_2(c)]\,, \tag{16}$$

where

$$g_1(c) = \int_a^c\int_a^b k(x; y)[f_1(y) - f_2(y)]dx\,dy$$

and

$$g_2(c) = \int_a^c \int_a^b k(x; y) f_2(y) dx\, dy \,.$$

Now Equation 16 may be rewritten

$$R_{n+1}(c) = [1 - \theta + \theta g_1(c)] R_n(c) + \theta g_2(c) \,,$$

and it follows easily from this difference equation that

$$R(c) = \lim_{n \to \infty} R_n(c) = \frac{g_2(c)}{1 - g_1(c)} \,,$$

which is precisely Equation 14, as was desired. Moreover, given Equation 14, Equation 13 follows by a simple argument from Equation 15. Q.E.D.

It is to be noted that Equation 14 has exactly the same form as Equation 5. More important, Equation 13 has the same form as Equation 9 for the noncontingent case, since

$$R(c) f_1(y) + [1 - R(c)] f_2(y)$$

is a proper density function. Consequently the same mathematical techniques apply to solving the integral equation for the smearing distribution.

More complicated contingent cases are easily constructed. For example, a straightforward generalization of the results just obtained is the division of the interval $[a, b]$ into an arbitrary finite number of subintervals rather than simply two. For this situation the experimenter selects r reinforcement distributions, one for each subinterval. Another example, further removed from the finite model, is the use by the experimenter of a continuous conditional reinforcement distribution $f(y \mid x)$. A conditional distribution of some interest would be one which, in the case of the circular disk mentioned earlier, would have a mean reinforcement some given number of degrees clockwise or counterclockwise from the response.

The *continuous contingent* case is formally defined by

$$f_n(y) = \int_a^b f(y \mid x) r_n(x) dx \,. \tag{17}$$

It is possible to establish the existence of the asymptotic response distribution for the continuous contingent case by considering this case as the limit of a properly selected sequence of simple contingent cases with an increasing number of subintervals. The argument in terms of upper and lower bounds is standard and yields the following theorem (we omit a detailed proof):

THEOREM. *For the continuous contingent case as defined by Equation* 17,

$$R(x) = \int_a^x \int_a^b \int_a^b k(x; y) f(y \mid t) r(t) dx\, dt\, dy \,.$$

Some Numerical Examples[3]

It should be useful to examine the form of the asymptotic response distributions when simple noncontingent reinforcement distributions are selected and it is assumed that the smearing distribution has some analytically elementary form. All of the examples to be considered refer to the circular disk apparatus mentioned above.

Example 1. *Parabolic Smearing Distribution.* We take as the density of the smearing distribution the parabola

$$k(x-y) = \begin{cases} \left(\frac{3}{4}\right)^{2/3} - (x-y)^2 & \text{for } |x-y| \leqq \left(\frac{3}{4}\right)^{1/3} \\ 0 & \text{for } \left(\frac{3}{4}\right)^{1/3} < |x-y| \leqq \pi . \end{cases} \tag{18}$$

For the reinforcement distribution, we use a simple linear density function whose maximum is at π:

$$f(y) = \begin{cases} \frac{y}{\pi^2} & \text{for } 0 \leqq y \leqq \pi \\ \frac{2}{\pi} - \frac{y}{\pi^2} & \text{for } \pi < y < 2\pi . \end{cases} \tag{19}$$

Moreover, for purposes of mathematical analysis we require that k and f be periodic functions with period 2π.

Our problem now is to use Equations 18 and 19 to find the density $r(x)$ of the asymptotic response distribution by integrating the derivative of the right side of Equation 9, namely,

$$r(x) = \int_0^{2\pi} k(x;y)f(y)\,dy = \int_0^{2\pi} k(x-y)f(y)\,dy . \tag{20}$$

The problem is more complex than the simple form of the functions k and f indicate, for the limits of integration must be carefully adjusted. What we are led to is a division of the circle into six regions with a different integrand for each. To begin with, since k and f are periodic with period 2π, we may replace $x-y$ by t and have instead of Equation 20,

$$r(x) = \int_0^{2\pi} k(t)f(x-t)\,dt . \tag{21}$$

But since $k(t)$ is 0 when $t^2 > a^2$, where

$$a = \left(\frac{3}{4}\right)^{1/3},$$

[3] I am indebted to Gordon Latta for several helpful suggestions in connection with the examples in this section.

we may for this example replace Equation 21 by

$$r(x) = \int_{-a}^{a} k(t)f(x-t)\,dt\ .$$

The six regions we have to consider are the following:

$$0 < x \leqq a, \qquad \pi - a < x \leqq \pi, \qquad \pi + a < x \leqq 2\pi - a\ ,$$
$$a < x \leqq \pi - a\ , \qquad \pi < x \leqq \pi + a\ , \qquad 2\pi - a < x \leqq 2\pi\ .$$

For the first one, $0 < x \leqq a$, we have to consider two cases: $t \geqq x$ and $t < x$. Thus we have for this region

$$r(x) = \int_{x}^{a} k(t)f(x-t)\,dt + \int_{-a}^{x} k(t)f(x-t)\,dt\ . \tag{22}$$

For both integrands of Equation 22, $k(t) = a^2 - t^2$, but in the case of $f(x-t)$, for the first one $t > x$ and therefore the argument $x - t$ is negative, which is not within the domain of definition of Equation 19; but given that f has period 2π, we have

$$f(x-t) = f(2\pi + x - t)$$

and $\pi < 2\pi + x - t < 2\pi$, whence from Equation 19

$$f(x-t) = \frac{2}{\pi} - \frac{(2\pi + x - t)}{\pi^2} = \frac{t-x}{\pi^2}\ .$$

For the second integrand of Equation 22, $f(x-t) = (x-t)/\pi^2$. Therefore, we obtain for $0 < x \leqq a$

$$\begin{aligned} r(x) &= \frac{1}{\pi^2}\int_{x}^{a}(a^2 - t^2)(t-x)\,dt + \frac{1}{\pi^2}\int_{-a}^{x}(a^2-t^2)(x-t)\,dt \\ &= \frac{1}{6\pi^2}(3a^4 + 6a^2x^2 - x^4)\ . \end{aligned}$$

We turn now to the second region, $a < x \leqq \pi - a$. Here $0 < x - t \leqq \pi$ and thus $f(x-t) = (x-t)/\pi^2$. Hence, for this region

$$r(x) = \frac{1}{\pi^2}\int_{-a}^{a}(a^2 - t^2)(x-t)\,dt = \frac{4}{3\pi^2}a^3x\ .$$

For the third region, $\pi - a < x \leqq \pi$, complications once again ensue. There are two cases to consider: $t \geqq x - \pi$ and $t < x - \pi$. If $t \geqq x - \pi$, then $x - t < \pi$ and $f(x-t) = (x-t)/\pi^2$. If $t < x - \pi$, then $x - t > \pi$ and

$$f(x-t) = \frac{2}{\pi} - \frac{x-t}{\pi^2}\ .$$

Thus, for $\pi - a < x \leqq \pi$,

$$r(x) = \frac{1}{\pi^2}\int_{x-\pi}^{a}(a^2 - t^2)(x-t)dt + \int_{-a}^{x-\pi}(a^2-t^2)\left(\frac{2}{\pi} - \frac{x-t}{\pi^2}\right)dt\ . \tag{23}$$

After integrating and simplifying, we get from Equation 23

$$r(x) = \frac{1}{\pi^2}\left[\frac{x^4}{6} - \frac{2\pi}{3}x^3 + (\pi^2 - a^2)x^2 + \left(2a^2\pi - \frac{2}{3}\pi^3\right)x + \frac{\pi^4}{6} - a^2\pi^2 + \frac{4}{3}\pi a^3 - \frac{a^4}{4}\right]$$

for $\pi - b < x \leqq \pi$.

Because $r(x)$ is symmetric about π, the other three regions have similar densities. More exactly, the interval $(\pi, \pi + a)$ corresponds to $(\pi - a, \pi)$, the interval $(\pi + a, 2\pi - a)$ to $(a, \pi - a)$, and the interval $(2\pi - a, 2\pi)$ to $(0, a)$. The graph of $r(x)$ from 0 to π is given in Fig. 1 below.

Example 2. Trigonometric Smearing Distribution. We take as the density of the smearing distribution the cosine function

$$k(x - y) = \frac{1}{4}\cos\frac{x - y}{2} \qquad \text{for} \quad -\pi < x - y < \pi . \tag{24}$$

For comparative purposes we retain Equation 19 as the reinforcement distribution. Because of the periodicity of k and f, we may, as is convenient for this example, replace Equation 21 by

$$r(x) = \int_{-\pi}^{\pi} k(t)f(x - t)\,dt . \tag{25}$$

To integrate Equation 25, using Equations 19 and 24, we need only decide how to break up the interval $(-\pi, \pi)$ according to the branch of $f(y)$ we are using. We may choose as our subintervals $(-\pi, x - \pi)$, $(x - \pi, x)$, (x, π), and consider only x, for $0 < x < \pi$.

For the first, we then have $-\pi < t < x - \pi$, whence, multiplying by -1 and adding x, we obtain $\pi + x > x - t > \pi$. Thus for this interval,

$$f(x - t) = \frac{2}{\pi} - \frac{x - t}{\pi^2} .$$

For the interval $(x - \pi, x)$, we have $x - \pi < t < x$, and consequently by the same argument as above $\pi - x > x - t > 0$. Thus for this interval,

$$f(x - t) = \frac{x - t}{\pi^2} .$$

Finally, for the third interval $x < t < \pi$, whence $0 > x - t > x - \pi$. Here $x - t$ is negative, and we add 2π to obtain $2\pi > 2\pi + x - t > \pi + x$. Hence for this interval,

$$f(x - t) = \frac{2}{\pi} - \frac{2\pi + x - t}{\pi^2} = \frac{t - x}{\pi^2} .$$

Applying these three results for f to Equation 25, we obtain for the asymptotic response density, for $0 < x < \pi$:

$$r(x) = \int_{-\pi}^{x-\pi}\left(\frac{1}{4}\cos\frac{t}{2}\right)\left(\frac{2}{\pi} - \frac{x-t}{\pi^2}\right)dt + \int_{x-\pi}^{x}\left(\frac{1}{4}\cos\frac{t}{2}\right)\frac{(x-t)}{\pi^2}\,dt$$
$$+ \int_{x}^{\pi}\left(\frac{1}{4}\cos\frac{t}{2}\right)\frac{(t-x)}{\pi^2}\,dt = \frac{1}{\pi^2}\left(2\sin\frac{x}{2} - 2\cos\frac{x}{2} + \pi - x\right). \tag{26}$$

Example 3. *Exponential Smearing Distribution.* For a final example, we take as the density of the smearing distribution the exponential function

$$k(x-y) = \frac{e^{-|x-y|}}{2(1-e^{-\pi})} \quad \text{for} \quad -\pi < x - y < \pi \,.$$

Again we use Equation 19 for the reinforcement distribution. The divisions of the interval $(-\pi, \pi)$ in Equation 26 are appropriate here, but in addition we must also divide the interval at 0, since

$$e^{-|t|} = \begin{cases} e^{-t} & \text{for} \quad t \geq 0 \\ e^{t} & \text{for} \quad t < 0 \,. \end{cases}$$

We thus have for the asymptotic response density function, for $0 < x < \pi$:

$$r(x) = \frac{1}{2\pi^2(1-e^{-\pi})}\left[\int_{-\pi}^{x-\pi} e^{t}(2\pi + t - x)\,dt + \int_{x-\pi}^{0} e^{t}(x-t)\,dt + \int_{0}^{x} e^{-t}(x-t)\,dt + \int_{x}^{\pi} e^{-t}(t-x)\,dt\right]$$
$$= \frac{-e^{x-\pi} + (x-\pi)e^{-\pi} + e^{-x} + x}{\pi^2(1-e^{-\pi})}\,.$$

The linear density function f of the reinforcement distribution and the three asymptotic response densities r_1, r_2, and r_3 of the three examples just given are plotted in Fig. 1 (r_1 is for the parabolic example, r_2 for the trigonometric, and r_3 for the exponential). The differences between r_1, r_2, and r_3 are sharp enough to expect experimental resolution of their relative validity. Actually, one of the most likely candidates for the smearing distribution is a truncated normal distribution with mean zero and variance to be estimated from the data. Such an example was not given here because the integrals cannot be given in closed form.

These three examples bring out the most striking difference between the finite and the continuous noncontingent models, namely, that the asymptotic response distribution is not in general identical with the reinforcement distribution; the obvious analogue of Equation 3 does not hold. It is to be anticipated that the exact form of the smearing distribution, and thus the exact relation between the reinforcement distribution and the asymptotic response distribution, will vary from one piece of apparatus to another. For example, in the case of the circular disk apparatus, it seems likely that when measurements are made in radians, the variance of the smearing distribution will vary inversely with the diameter of the disk.

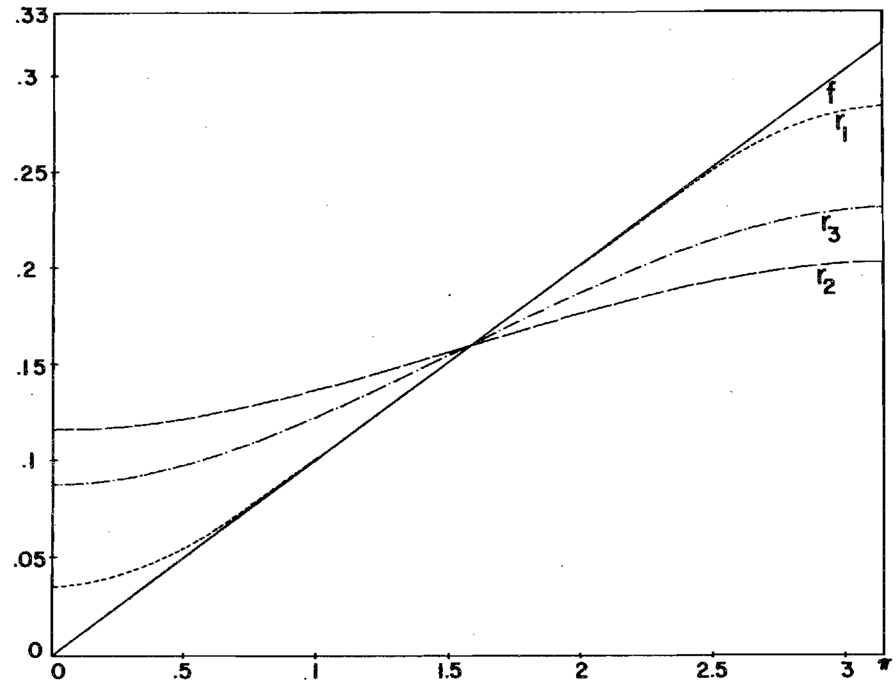

FIGURE 1. Asymptotic response distribution for linear reinforcement and three different smearing distributions.

Further Results for the Noncontingent Case

For the noncontingent case of the continuous model we may without much difficulty derive results analogous to those in Chapter 8, Sec. 8. To begin with, analogous to the moments 5.1 of Chapter 8, for a subinterval (a_1, a_2) of $[a, b]$, we have

(27)
$$V^{v}_{a_1,a_2,n} = \int_{a_1}^{a_2}\int_{s_{n-1}} j^{v}_{n}(x \mid s_{n-1})\, j(s_{n-1})\, dx\, ds_{n-1} \,,$$

where integration over s_{n-1} means integration over the whole interval $[a, b]$ for each of the $2(n-1)$ random variables corresponding to the sequence s_{n-1}. Note that the subscript n is omitted on the second density j, a practice which is followed in the sequel when a subscript is not needed for clarity.

A general recursion for the moments just defined, analogous to Equation 8.1 of Chapter 8, is obtainable, but because of its length, we restrict ourselves here to $v = 2$, which corresponds to equation 8.4 of Chapter 8. We emphasize again that these results are for the noncontingent case, that is, when $f_n(y) = f(y)$ for all n.

Theorem.

(28) $$V^2_{a_1,a_2,n+1} = (1-\theta)^2 V^2_{a_1,a_2,n} + 2\theta(1-\theta)\int_{a_1}^{a_2}\int_a^b r_n(x)k(x;y)f(y)\,dx\,dy + \theta^2\int_{a_1}^{a_2}\int_a^b k^2(x;y)f(y)\,dx\,dy\,.$$

Proof. By virtue of Equation 27,

$$V^2_{a_1,a_2,n+1} = \int_{a_1}^{a_2}\int_{s_n} j^2_{n+1}(x \mid s_n)\,j(s_n)\,dx\,ds_n\,,$$

but each $s_n = (y_n, x_n, s_{n-1})$, whence

$$V^2_{a_1,a_2,n+1} = \int_{a_1}^{a_2}\int_a^b\int_a^b\int_{s_{n-1}} j^2_{n+1}(x \mid y_n, x_n, s_{n-1})\,j(y_n, x_n, s_{n-1})\,dx\,dy_n\,dx_n\,ds_{n-1}\,.$$

Applying Axiom C3 to the right-hand side of the above equation,

$$\begin{aligned} V^2_{a_1,a_2,n+1} &= \int_{a_1}^{a_2}\int_a^b\int_a^b\int_{s_{n-1}} [(1-\theta)\,j_n(x \mid s_{n-1}) \\ &\qquad + \theta k(x;y_n)]^2\,j(y_n, x_n, s_{n-1})dx\,dy_n\,dx_n\,ds_{n-1} \\ &= \int_{a_1}^{a_2}\int_{s_{n-1}} (1-\theta)^2 j^2_n(x \mid s_{n-1})\,j(s_{n-1})\,dx\,ds_{n-1} \\ &\qquad + 2\theta(1-\theta)\int_{a_1}^{a_2}\int_a^b\int_{s_{n-1}} j_n(x \mid s_{n-1})k(x;y_n)f(y_n)\,j(s_{n-1})\,dx\,dy_n\,ds_{n-1} \\ &\qquad + \theta^2\int_{a_1}^{a_2}\int_a^b k^2(x;y_n)f(y_n)\,dx\,dy_n\,, \end{aligned}$$

and the theorem easily follows from this last equation, since the first term on the right is simply $(1-\theta)^2 V^2_{a_1,a_2,n}$, and we may integrate out s_{n-1} in the second term on the right and thus replace $j_n(x \mid s_{n-1})\,j(s_{n-1})$ in the integrand by the mean response density $r_n(x)$. Q.E.D.

Asymptotically, we have from Equation 28, corresponding to the expression following Definition 8.3 of Chapter 8,

$$V^2_{a_1,a_2} = \frac{2(1-\theta)}{2-\theta}\int_{a_1}^{a_2}\int_a^b r(x)k(x;y)f(y)\,dx\,dy + \frac{\theta}{2-\theta}\int_{a_1}^{a_2}\int_a^b k^2(x;y)f(y)\,dx\,dy\,.$$

To obtain an expression corresponding to the doublet term $P(A_{1,n+1} \cap A_{1,n})$, we need first to define some cross-moments similar to the moments $V^v_{a_1,a_2,n}$. Note that the subscripts of x in this definition do not refer to trial numbers but index the v variables required for the definition:

(29) $$W^v_{a_1,a_2,n} = \int_{a_1}^{a_2}\cdots\int_{a_1}^{a_2}\int_{s_{n-1}} j_n(x_1 \mid s_{n-1})\cdots j_n(x_v \mid s_{n-1})\,j(s_{n-1})\,dx_1\cdots dx_v\,ds_{n-1}\,.$$

Using the notation $j_{n+1,n}(x', x)$ for the joint distribution of responses on trials $n+1$ and n, we then have the following theorem.

THEOREM. *For* $a \leqq a_1 < a_2 \leqq b$

$$\int_{a_1}^{a_2}\int_{a_1}^{a_2} j_{n+1,n}(x', x)\,dx'\,dx = (1-\theta)\,W^2_{a_1,a_2,n} + \theta[R_n(a_2) - R_n(a_1)]\int_{a_1}^{a_2}\int_a^b k(x; y)f(y)\,dx\,dy\,. \tag{30}$$

PROOF.

$$\int_{a_1}^{a_2}\int_{a_1}^{a_2} j_{n+1,n}(x', x)\,dx'\,dx = \int_{a_1}^{a_2}\int_a^b\int_{a_1}^{a_2}\int_{s_{n-1}} j_{n+1}(x' \mid y, x, s_{n-1})f(y)j_n(x \mid s_{n-1})j(s_{n-1})\,dx'\,dy\,dx\,ds_{n-1}\,,$$

where we make use on the right-hand side of the noncontingency of reinforcement. Applying Axiom C3, we have that the right-hand side is

$$\int_{a_1}^{a_2}\int_a^b\int_{a_1}^{a_2}\int_{s_{n-1}} [(1-\theta)j_n(x' \mid s_{n-1}) + \theta k(x'; y)]f(y)j_n(x \mid s_{n-1})j(s_{n-1})\,dx'\,dy\,dx\,ds_{n-1}\,.$$

Using Equation 29, and integrating out s_{n-1} in the second term, we get

$$(1-\theta)\,W^2_{a_1,a_2,n} + \theta\int_{a_1}^{a_2}\int_a^b\int_{a_1}^{a_2} k(x'; y)f(y)r_n(x)\,dx'\,dy\,dx\,.$$

Finally, integrating out x in the second term, we obtain the desired result. Q.E.D.

The similarity of Equation 30 to the corresponding expression in Chapter 8,

$$(1-\theta)V^2_n + \theta\pi V^1_n\,,$$

should be emphasized. $W^2_{a_1,a_2,n}$ corresponds to the second moment V^2_n; the probability $R_n(a_2) - R_n(a_1)$ of responding in the interval (a_1, a_2) to the first moment V^1_n; and the integral to the probability π.

Using the techniques of proof for the preceding two theorems it is not difficult, though somewhat tedious, to get analogues of Equations 8.17 (the covariance term) and the serial correlation r_m of Chapter 8. But since these analogues involve rather lengthy integral expressions in terms of the unspecified smearing and reinforcement distributions, we omit them here.

In conclusion, we obtain some relatively simple expressions for the expected value and variance (asymptotically) of the response random variable A_n. The expectation of A_n is defined by

$$E(A_n) = \int_a^b x r_n(x)\,dx\,.$$

Applying the usual techniques to the right-hand side and using Axiom C3, we have

$$\int_a^b x r_n(x)\,dx = \int_a^b\int_a^b\int_a^b\int_{s_{n-2}} x j_n(x \mid y_{n-1}, x_{n-1}, s_{n-2})\, j(y_{n-1}, x_{n-1}, s_{n-2}) \cdot dx\, dy_{n-1} dx_{n-1} dx_{n-2}$$

$$= \int_a^b\int_a^b\int_{s_{n-2}} x[(1-\theta) j_{n-1}(x \mid s_{n-2}) + \theta k(x; y_{n-1})] \cdot j(y_{n-1}, s_{n-2})\, dx\, dy_{n-1}\, ds_{n-2}$$

$$= (1-\theta)\int_a^b x r_{n-1}(x)\,dx + \int_a^b\int_a^b x k(x; y) f(y)\,dx\,dy\ .$$

Now the first term on the right of the last expression is simply $E(A_{n-1})$, and thus we have the following recursion.

THEOREM.

$$E(A_n) = (1-\theta)E(A_{n-1}) + \theta\int_a^b\int_a^b x k(x; y) f(y)\,dx\,dy\ .$$

Moreover, asymptotically

$$E(A_\infty) = \lim_{n\to\infty} E(A_n) = \int_a^b\int_a^b x k(x; y) f(y)\,dx\,dy\ .$$

It is not difficult to show that for the three numerical examples considered in the preceding section, $E(A_\infty) = \pi$.

The recurrence relation for the variance of A_n is not simple, but asymptotically we have

$$\text{(31)}\qquad \operatorname{var}(A_\infty) = \int_a^b\int_a^b x^2 k(x; y) f(y)\,dx\,dy - \left[\int_a^b\int_a^b x k(x; y) f(y)\,dx\,dy\right]^2 .$$

If it is assumed, for the circular disk or similar apparatus, that the smearing distribution is a truncated normal distribution with mean zero and unknown variance, then Equation 31 may be used to estimate (somewhat inefficiently) this variance, for $\operatorname{var}(A_\infty)$ may be computed directly from the empirical histogram of asymptotic response frequencies.