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A PARTICLE THEORY OF THE CASIMIR EFFECT

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In previous works, Suppes and de Barros used a pure particle model to derive interference effects, where individual photons have well-defined trajectories, and hence no wave properties. In the present paper we extend that description to account for the Casimir effect. We consider that the linear momentum $\sum \frac{1}{2}\hbar\mathbf{k}$ of the vacuum state in quantum electrodynamics corresponds to the linear momentum of virtual photons. The Casimir effect, in the cases of two parallel plates and the solid ball, is explained in terms of the pressure caused by the photons. Contrary to quantum electrodynamics, we assume a finite number of virtual photons.

Key words: photon, trajectories, QED, Casimir effect, quantum vacuum.

1. INTRODUCTION

Suppes and de Barros (1994a, 1994b, 1996) began a founda-

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tional analysis of diffraction of light which formulated a probabilistic theory of photons with well-defined photon trajectories and without wave properties. The wave properties come from the expectation density of the photons. The previous works about this probability model and the present paper can be understood as an attempt to replace the quantum theory of the electromagnetic field. We do not know if our model can produce all expected results of quantum electrodynamics (QED). Only further investigations with other phenomena such as the Lamb shift, the anomalous magnetic moment of the electron, nonexistence of Bell-type hidden variables, etc, can answer this question. The photons are also regarded as virtual, because they are not directly observable, including their annihilation of each other (see assumptions below). What can be detected is the interaction with matter. The meaning of *virtual* used here is not the same as in QED. In summary, our assumptions are:

- Photons are emitted by harmonically oscillating sources;
- They have definite trajectories;
- They have a probability of being scattered at a slit;
- Absorbers, like sources, are periodic;
- Photons have positive and negative states (+-photons and −-photons) which locally interfere, when being absorbed;
- Photons change their states when reflected by a perfect conductor.

These assumptions are meant to be more or less natural for particles. From these assumptions we can define fields as we do below, as a purely probabilistic concept, without additional physical meaning. Consequently, the probabilistic properties of the defined field are derived in a manner familiar from stochastic processes from the properties of collections of the sample paths, i.e., trajectories of photons. It remains to be seen whether this reduction of fields to distributions of particles with linear trajectories can be carried through for all parts of QED, as discussed above. In the case of the Casimir effect, our particle model is used to calculate pressure exerted on the plates in the spirit of the particle theory of classical statistical mechanics.

The expected density of $\pm$-photons emitted at t in the interval dt is given by

$$s_{\pm}(t) = \frac{A_s}{2}(1 \pm \cos \omega t), \tag{1}$$

where ω is the frequency of a harmonically oscillating source, A_s is a constant determined by the source, and t is time. If a photon is emitted at t', $0 \leq t' \leq t$, then at time t the photon has traveled (with speed c) a distance r, where

$$t - t' = \frac{r}{c}. \tag{2}$$

The conditional space-time expectation density of $\pm$-photons for a spherically symmetric source with given periodicity ω is

$$h_{\pm}(t, r|\omega) = \frac{A}{8\pi r^2}\left(1 \pm \cos\omega\left(t - \frac{r}{c}\right)\right), \tag{3}$$

where A is a real constant.

The scalar field defined in terms of the expectation density $h_{\pm}(t, r|\omega)$ is

$$\mathcal{E} = \mathcal{E}_0 \frac{h_+ - h_-}{\sqrt{h_+ + h_-}}, \tag{4}$$

where $\mathcal{E}_0$ is a scalar physical constant. Using (3), (4) may be rewritten for a spherically symmetric source as

$$\mathcal{E} = \mathcal{E}_0 \sqrt{\frac{A}{4\pi r^2}} \cos\omega\left(t - \frac{r}{c}\right). \tag{5}$$

In the present paper we construct a corpuscular model for the Casimir effect, following the ideas previously proposed by Suppes and de Barros (1994a, 1994b, 1996). First, we study the case of the parallel plates, and then the solid ball.

2. QUANTUM VACUUM

A pure particle theory must postulate properties of the quantum vacuum if standard field-theoretic methods of computing the Casimir effect, and similar phenomena such as the Lamb shift, are to be closely approximated.

In QED the vacuum state has a zero-point energy $\frac{1}{2}\hbar\omega$ and a linear momentum $\frac{1}{2}\hbar\mathbf{k}$, which are derived from second quantization. Here we introduce directly, as a first postulate, that $\frac{1}{2}\hbar\mathbf{k}$ corresponds to the linear momentum of one virtual photon. The elementary assumptions introduced in Sec. 1 are not strong enough to support a derivation of this postulate. Our second postulate is that we have a distribution function of k, $f(k)$, which is a probability density for the distribution of photons with respect to k. Our third postulate establishes that both $+$-photons and $-$-photons contribute to the delivery of linear momentum on a reflective surface. So, the conditional expectation density of photons, given k, that strike a point on the conductor surface is

$$h(t, r_S|k) = h_+(t, r_S|k) + h_-(t, r_S|k), \tag{6}$$

where r_S is a surface point.

3. PARALLEL PLATES

We consider here the case of two perfectly conducting parallel plates, standing face to face in vacuum at a distance d much smaller than their lateral extensions. It is well known that such plates attract each other with a force per unit area (pressure) due to the vacuum energy, as predicted by Casimir (1948), given by

$$P = -\frac{\pi^2 \hbar c}{240 d^4}. \tag{7}$$

Usually, such an attraction is explained in terms of the vacuum field. We use a random distribution of oscillating sources of photons, in the vacuum, which do not interfere with each other, to derive (7) in our conceptual framework.

The photons outside the plates that strike such surfaces act to push the plates together, while reflections of the photons confined between the plates push them apart. This idea is proposed by Milonni, Cook, and Goggin (1988) and also presented in (Milonni, 1994), but not actually developed from a pure particle viewpoint.

The photons that we are considering must satisfy a probability density $f(k) \geq 0$. Rather than assume an explicit expression for $f(k)$ (which requires some assumptions about the virtual photons), we prefer to state some properties that $f(k)$ must satisfy:

(i) $\int_0^\infty \int_0^\infty \int_0^\infty f(k) dk_x dk_y dk_z = 1$, and the mean and the variance of $f(k)$ are finite;
(ii) There exists a constant H such that $h(t, r_S|k) < H$.
(iii) $h(t, r_S|k) f(k)|_{k=0} = 1$, and all derivatives of this expression vanish at $k_z = 0$.

From (i)~(iii) and assumptions made earlier, we may infer, contrary to a standard result of QED, that in our theory the number of virtual photons is finite for any bounded region of space-time. We also infer that $h(t, r_S|k) f(k)|_{k=\infty} = 0$, which is intuitively an expected property of a cutoff function. We note that $h(t, r_S|k)$ and $f(k)$ are not identified as specific functions. We have generalized from standard cutoff functions, such as an exponential function, to give reasonable sufficient conditions that many different functions satisfy. We do not know enough about the quantum vacuum to derive a particular choice.

We divide the xyz space into parallelepipeds of sides L_x, L_y, and L_z, as in the usual description of QED. So, all k_x, k_y, and k_z must assume discrete values, as is explained in the next paragraphs.

We note that when reflected a photon changes its state from positive to negative and vice versa. This single change for perfect conductors implies that the defined scalar field, given by (4), vanishes at the reflecting surface. For further details see (Suppes and de Barros, 1996). So, according to (5) and recalling that $k = \omega/c$, we have at the wall:

$$\cos\left(\omega t - \omega\frac{r}{c}\right) = \cos(\omega t - kr) = 0. \tag{8}$$

If we set $\omega t = \pi/2$, which corresponds to a convenient choice for the origin of time, it is easy to see that the values of k_x, k_y, and k_z that satisfy the boundary condition in $x = L_x$, $y = L_y$, and $z = L_z$ are:

$$\frac{k_{x,y,z}}{\pi} = \frac{n}{L_{x,y,z}}. \tag{9}$$

But the condition that the scalar electric field vanishes at the surface of the reflecting walls is not sufficient to explain the periodicity given by (9). A natural question arises: what about the photons with linear momenta that do not satisfy (9)? We recall that reflectors, like absorbers (Suppes and de Barros, 1994b), behave periodically, since the photons are continuously hitting the plates. Thus, the probability of reflecting a photon is given by:

$$p = C(1 + \cos(\omega t + \psi)), \tag{10}$$

where ψ is a certain phase. If $p = 0$, then there is no interaction with the plates, which means that no momentum is delivered to it.

As an example, consider the first strike of a photon on a plate perpendicular to the z axis. Such a surface is not oscillating before the strike. But after reflection, the wall oscillates with the same frequency ω associated to the linear momentum $k = \omega/c$. The particle reflects on the other wall and returns to the first wall with a phase $2L_zk_z$. But we must have $2L_zk_z = 2n\pi$, from (10), if the particle is to be reflected again on its return to the first wall. Obviously, $\cos(\omega t - 2L_zk_z) = \cos(\omega t)$ if and only if $2L_zk_z = 2n\pi$.

If we consider $L_{x,y,z}$ very large compared with any physical dimensions of interest, we can assume that the $k_{x,y,z}$ approach a continuum. This is what holds for photons outside the plates.

Now we start to derive the pressure on the plates. We begin with the inward pressure. The expected number of photons that strike the area dS of one of the plates, i.e., have trajectories in the direction of the plates, within the time interval dt is

$$h(t, r_S|k)f(k)\frac{1}{\pi^3}dk_x dk_y dk_z \cos\gamma\, c\, dt\, dS, \tag{11}$$

where γ is the angle of incidence of the photons on the plate with respect to the normal of the surface, i.e., $\cos\gamma = k_z/k$, where $k = \sqrt{k_x^2 + k_y^2 + k_z^2}$. Thus, the element of volume that we are taking into account is $\cos\gamma\, c\, dt\, dS$. The factor $\frac{1}{\pi^3}$ is justified by (9), since outside the plates we approach the continuum as a limit.

The momentum delivered to the plate by a single reflected photon is equal to the negative of the change in the momentum of the photon. In other words the momentum is equal to $2\frac{1}{2}\hbar k_z$, if we consider the plate perpendicular to the z component of the xyz system of coordinates. Therefore, the expected linear momentum transfered to an area dS on the plate during the time interval dt is

$$\frac{\hbar}{\pi^3}\frac{k_z^2}{k}h(t, r_S|k)f(k)dk_x dk_y dk_z c\, dt\, dS. \tag{12}$$

The force on the plate is obtained by dividing (12) by dt. The pressure is obtained by dividing the force by dS. We denote the inward pressure as P_{in} and the outward pressure as P_{out}. Hence:

$$dP_{in} = \frac{\hbar c}{\pi^3}\frac{k_z^2 h(t, r_S|k)f(k)}{\sqrt{k_x^2 + k_y^2 + k_z^2}}dk_x dk_y dk_z. \tag{13}$$

Integrating over momentum:

$$P_{in} = \frac{\hbar c}{\pi^3}\int_0^\infty dk_x \int_0^\infty dk_y \int_0^\infty dk_z \frac{h(t, r_S|k)f(k)k_z^2}{\sqrt{k_x^2 + k_y^2 + k_z^2}}. \tag{14}$$

The equation given above is identical to a result due to Milonni, Cook and Goggin (1988), if we consider that $h(t, r_S|k)f(k)$ has the role of the usual cutoff function.

To obtain the expression of the outward pressure we use similar arguments. But now, because of the small distance d between the plates, we must take into account the periodicity given in (9), at least for the z component. Hence:

$$P_{out} = \frac{\hbar c}{\pi^2 d} \sum_{n=1}^{\infty} \int_0^{\infty} dk_x \int_0^{\infty} dk_y \frac{h(t, r_S|k) f(k) (\frac{n\pi}{d})^2}{\sqrt{k_x^2 + k_y^2 + (\frac{n\pi}{d})^2}}. \tag{15}$$

We note that it follows from (i)~(iii) that P_{in} and P_{out} are both finite.

The resultant pressure is given by:

$$\begin{aligned} &P_{out} - P_{in} \\ =&\frac{\pi^2 \hbar c}{4d^4} \sum_{n=1}^{\infty} n^2 \int_0^{\infty} dx \frac{h(t, r_S|x, u) f(\sqrt{x + n^2})}{\sqrt{x + n^2}} - \\ &\frac{\pi^2 \hbar c}{4d^4} \int_0^{\infty} du\, u^2 \int_0^{\infty} dx \frac{h(t, r_S|x, u) f(\sqrt{x + u^2})}{\sqrt{x + u^2}}, \end{aligned} \tag{16}$$

where, by change of variables,

$$f(k) = f(\sqrt{x + u^2}), \quad x = x'^2 = \frac{k_x^2 d^2}{\pi^2} + \frac{k_y^2 d^2}{\pi^2},$$
$$u = k_z \frac{d}{\pi}, \; \theta = \tan\left(\frac{k_y}{k_x}\right), \quad dk_x dk_y = x' dx' d\theta \frac{\pi^2}{d^2}$$

The expression $h(t, r_S|x, u) f(\sqrt{x + u^2})$ corresponds to a cutoff function. In our model, $h(t, r_S|k)$ is bounded and $f(\sqrt{x + u^2})$ has the physical interpretation of a probability density of the frequencies of the photons.

Frequently it is assumed that the cutoff function has the property of going to zero as k approaches infinity and going to one when k approaches zero. This is justified physically with the hypothesis that the conductivity of the reflecting conductors decreases to zero as the frequency gets high. Since $h(t, r_S|k)$ is bounded, it is easy to see that the product $h(t, r_S|k) f(k)$ must assume a similar role with respect to the cutoff, from a mathematical standpoint. According to the assumptions that we made about $h(t, r_S|k)$ and $f(k)$, all the properties of a cutoff function are satisfied for $h(t, r_S|k) f(k)$.

If we consider

$$F(u) = u^2 \int_0^{\infty} dx \frac{h(t, r_S|x, u) f(\sqrt{x + u^2})}{\sqrt{x + u^2}}, \tag{17}$$

it is clear that the Euler-MacLaurin summation formula (Abramowitz and Stegun, 1971) may be applied to (16). Contrary to the standard QED treatment, we apply this formula to a convergent rather than a divergent series. Therefore, the factor that is multiplying $\frac{\pi^2 \hbar c}{4d^4}$ in (16) may be written as:

$$\sum_{n=1}^{\infty} F(n) - \int_0^{\infty} du F(u) = -\frac{1}{2}F(0) - \frac{1}{12}F'(0) + \frac{1}{720}F'''(0)... \quad (18)$$

for $\lim_{u\to\infty} F(u) = 0$, since $\sum_{n=1}^{\infty} F(n)$ is finite and so $\lim_{n\to\infty} F(n) = 0$. We note that $F(0) = 0$, $F'(0) = 0$, $F'''(0) = -12h(t, r_S|0)$, and all higher derivatives $F^{(n)}(0)$, where n is odd, vanish in accordance with assumption (iii), as is shown in the appendix. Since by (iii) $h(t, r_S|0)f(0) = 1$,

$$P_{out} - P_{in} = -\frac{\pi^2 \hbar c}{240 d^4}. \quad (19)$$

Equation (19) is identical to (7), which completes our derivation of the Casimir effect for parallel plates.

4. THE SOLID BALL

Many authors have considered the case of the Casimir force for solid balls and cavities like spheres, hemispheres, and spheroids. Balian and Duplantier (1977) developed a method that establishes an expansion for the Green functions describing electromagnetic waves in the presence of a perfectly conducting boundary. Later, they applied their method to the study of the Casimir free energy of the electromagnetic field in regions bounded by thin perfect conductors with arbitrary shape (Balian and Duplantier, 1978). Brevik and Einevoll (1988) used Schwinger's source theory to establish the Casimir surface force in the case of a solid ball, considering $\epsilon(\omega)\mu(\omega) = 1$, where $\epsilon(\omega)$ is the spectral permittivity and $\mu(\omega)$ is the spectral permeability. In 1990, Brevik and Sollie (1990) calculated the Casimir surface force on a spherical shell, assuming the same condition $\epsilon(\omega)\mu(\omega) = 1$. Barton (1991a, 1991b) uses standard statistical and quantum physics to analyze the fluctuations of the Casimir stress exerted on a flat perfect conductor by the vacuum electromagnetic fields in adjacent space. Eberlein (1992) makes an extension of Barton's work, calculating the mean-square forces acting on spheres and hemispheres of variable sizes.

We adopt here the same corpuscular model presented in last section, to the case of a solid ball of radius a surrounded by vacuum.

Consider a ball, centered at the origin of a spherical coordinate system (ρ, φ, θ). At each point of the surface of the ball, we define a Cartesian system of coordinates, with axes $\perp$, $\|_1$, and $\|_2$. $\perp$ is the normal to the surface, and $\|_1$ and $\|_2$ are tangent to the sphere.

As in the case of the parallel plates, we assume a distribution function $f(k)$ satisfying the same properties assumed in the last section. The expected number of photons that strike the area $dS = a^2 \sin\varphi \, d\varphi \, d\theta$ on the surface of the solid ball, within the time interval dt is

$$h(t, r_S|k) f(k) dk_\perp dk_{\|_1} dk_{\|_2} c \, dt \; \cos\gamma \, a^2 \sin\varphi \, d\varphi \, d\theta, \tag{20}$$

where γ is the angle of incidence of the photons with respect to the normal of the surface, i.e., $\cos\gamma = k_\perp / k$, and $k = \sqrt{k_\perp^2 + k_{\|_1}^2 + k_{\|_2}^2}$.

As in the case of the plates, the momentum delivered to the ball by a single reflecting photon is $2\frac{1}{2}\hbar k_\perp$. The linear momentum on the ball is

$$\frac{\hbar k_\perp^2 h(t, r_S|k) f(k)}{\pi^3 k} dk_\perp dk_{\|_1} dk_{\|_2} c \, dt \; a^2 \sin\varphi \, d\varphi \, d\theta. \tag{21}$$

The force is obtained by dividing the expression above by dt:

$$dF = \frac{\hbar c}{\pi^3} a^2 h(t, r_S|k) f(k) \frac{k_\perp^2}{\sqrt{k_\perp^2 + k_{\|_1}^2 + k_{\|_2}^2}} dk_\perp dk_{\|_1} dk_{\|_2} \sin\varphi \, d\varphi \, d\theta. \tag{22}$$

Integrating:

$$F = \frac{4a^2 \hbar c}{\pi^2} \int_0^\infty dk_\perp \int_0^\infty dk_{\|_1} \int_0^\infty dk_{\|_2} \frac{h(t, r_S|k) f(k) k_\perp^2}{\sqrt{k_\perp^2 + k_{\|_1}^2 + k_{\|_2}^2}}. \tag{23}$$

By the arguments used earlier, the force F is finite.

Our result depends explicitly on $h(t, r_S|k) f(k)$ in (23), which has, as in the case of the parallel plates, a role similar to a cutoff. This is a consequence of the geometry of the problem. Brevik and Einevoll (1988) obtained another expression for the Casimir force in the case of a solid ball, which directly depends on a typical value $(3 \times 10^6 \text{sec}^{-1})$ for the cutoff frequency.

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APPENDIX

It follows from (17) that

$$F(u) = u^2 \int_0^\infty dx \frac{g(\sqrt{x+u^2})}{\sqrt{x+u^2}} = 2u^2 \int_0^\infty dx\, g(\sqrt{x+u^2})\frac{1}{2}(x+u^2)^{-1/2}, \tag{24}$$

where $g(\sqrt{x+u^2}) = h(t, r_S|x,u)f(\sqrt{x+u^2})$. If we put $y = (x + u^2)^{1/2}$, then

$$F(u) = 2u^2 \int_u^\infty dy\, g(y). \tag{25}$$

Before the evaluation of the derivatives of $F(u)$, we must observe that

$$\frac{d}{du}\int_u^\infty dy\, g(y) = -g(u), \tag{26}$$

since $g(\infty) = 0$. Hence:

$$F'(u) = 4u \int_u^\infty dy\, g(y) - 2u^2 g(u), \tag{27}$$

$$F'''(u) = -12g(u) - 12ug'(u) - 2u^2 g''(u), \tag{28}$$

and all higher derivatives $F^{(2n+1)}(u)$ vanish at $u = 0$ if the even derivatives of $g(u)$ vanish at the same point, which is assumed in (iii).

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