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AXIOMS FOR RELATIVISTIC KINEMATICS WITH OR WITHOUT PARITY

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1. **Introduction.** The primary aim of this paper is to give an elementary derivation of the Lorentz transformations, without any assumptions of continuity or linearity, from a single axiom concerning invariance of the relativistic distance between any two space-time points connected by an inertial path. The concluding section considers extensions of the theory of relativistic kinematics which will destroy conservation of temporal parity, that is, extensions which are not invariant under time reversals.

It is philosophically and empirically interesting that the Lorentz transformations can be derived without any extraneous assumptions of continuity or differentiability. In a word, the single assumption needed for relativistic kinematics is that all observers at rest in inertial frames get identical measurements of relativistic distances along inertial paths when their measuring instruments have identical calibrations. Note that it is a consequence and *not* an assumption that these observers are moving with a uniform velocity with respect to each other. Granted the possibility of perfect measurements everywhere of relativistic intervals, this single axiom isolates in a precise way the narrow operational basis needed for the special theory of relativity.

Prior to any search of the literature it would seem that this result would be well-known, but I have not succeeded in finding the proof anywhere. Every physics textbook on relativity makes a linearity assumption at the minimum. In geometrical discussions of indefinite quadratic forms it is often remarked that the relativistic interval is invariant under the Lorentz group, but it is not proved that it is invariant under no wider group, which is the main fact established here. Some further remarks in this connection are made at the end of Section 2.

2. **Primitive Notions and Single Axiom.** Our single initial axiom for relativistic kinematics is based on three primitive notions, each of which has a simple physical interpretation. The first notion is an arbitrary set X

interpreted as the set of *physical space-time points*. The second notion is a non-empty family $\mathfrak{F}$ of one-one functions mapping X onto R_4, the set of all ordered quadruples of real numbers. (Thus X must have the power of the continuum.) Intuitively each function in $\mathfrak{F}$ represents an *inertial space-time frame of reference*, or, more explicitly, a space-time measuring apparatus at rest in an inertial frame. If $x \in X$, $f \in \mathfrak{F}$, and $f(x) = \langle x_1, x_2, x_3, t\rangle$ then x_1, x_2, and x_3 are the three orthogonal spatial coordinates of the point x, and t the time coordinate, with respect to the frame f. For a more explicit formal notation, $f_i(x)$ is the ith coordinate of the space-time point x with respect to the frame f, for $i = 1, \ldots, 4$. The third primitive notion is a positive number c, which is to be interpreted as the *speed of light.*

It is convenient to have a notation for the *relativistic distance* with respect to a frame f between any two space-time points x and y.

DEFINITION 1. *If $x, y \in X$ and $f \in \mathfrak{F}$ then*

$$I_f(xy) = \sqrt{\sum_{i=1}^{3} [f_i(x) - f_i(y)]^2 - c^2[f_4(x) - f_4(y)]^2}.$$

(We always take the square-root with positive sign.) If f is an inertial frame, then (i) $I_f(xy) = 0$ if x and y are connected by a light line, (ii) $I_f^2(xy) < 0$ if x and y lie on an inertial path (the square is negative since $I_f(xy)$ is imaginary); (iii) $I(xy) > 0$ if x and y are separated by a "space-like" interval. We use (ii) for a formal definition.

DEFINITION 2. *If $x, y \in X$ and $f \in \mathfrak{F}$ then* x AND y LIE ON AN INERTIAL PATH WITH RESPECT TO f *if and only if* $I_f^2(xy) < 0$.

It will also occasionally be useful to characterize inertial paths in terms of their speed. We may do this informally as follows. By the *slope* of a line α in R_4, whose projection on the 4th coordinate (the time coordinate) is a non-degenerate segment, we mean the three-dimensional vector W such that for any two distinct points $\langle Z_1, t_1\rangle$ and $\langle Z_2, t_2\rangle$ of α

$$W = \frac{Z_1 - Z_2}{t_1 - t_2}.$$

By the *speed* of α we mean the non-negative number $|W|$. An *inertial path* is a line in R_4 whose speed is less than c; and a *light line* is of course a line whose speed is c.

The single axiom we require is embodied in the following definition.

DEFINITION 3. *A system* $\mathfrak{X} = \langle X, \mathfrak{F}, c\rangle$ *is a* COLLECTION OF RELATIVISTIC FRAMES *if and only if for every x, y in X, whenever x and y lie on an inertial path with respect to some frame in* $\mathfrak{F}$, *then for all f, f' in* $\mathfrak{F}$

$$(1) \qquad I_f(xy) = I_{f'}(xy).$$

I orginally formulated this invariance axiom so as to require that equation (1) hold for *all* space-time points x and y ,that is, without restricting them to lie on an inertial path (with respect to some frame in $\mathfrak{F}$). Walter Noll pointed out to me that with this stronger axiom no physically motivated arguments of the kind given below are required to prove that any two frames in $\mathfrak{F}$ are related by a linear transformation; a relatively simple algebraic argument may be given to show this.

On the other hand, when the invariance assumption is restricted, as it is here, to distances between points on inertial paths, the line of argument formalized in the theorems of the next section seems necessary. This restriction to pairs of points on inertial paths is physically natural because their distances $I_f(xy)$ are more susceptible to direct measurements than are the distances of points separated by a space-like interval (i.e., $I_f(xy) > 0$).

3. **Theorems.** In proving the main result that any two frames in $\mathfrak{F}$ are related by a Lorentz transformation, some preliminary definitions, theorems and lemmas will be useful. We shall use freely the geometrical language appropriate to Euclidean four-dimensional space with the ordinary positive definite quadratic form.

THEOREM 1. *If* $k \geq 0$ *and* $f(x) - f(y) = k[f(u) - f(v)]$ *then* $I_f(xy) = kI_f(uv)$.

PROOF: If $k = 0$, the theorem is immediate. So we need to consider the case for which $k > 0$. It follows from the hypothesis of the theorem that

$$(1) \qquad x_i - y_i = k(u_i - v_i) \text{ for } i = 1, \ldots, 4,$$

where, for brevity here and subsequently, when we are considering a fixed element f of $\mathfrak{F}$, $f_i(x) = x_i$, etc. Using (1) and Definition 1 we then have:

$$I_f(xy) = \sqrt{\sum_{i=1}^{3} (x_i - y_i)^2 - c^2(x_4 - y_4)^2}$$
$$= \sqrt{\sum_{i=1}^{3} k^2(u_i - v_i)^2 - c^2k^2(u_4 - v_4)^2}$$
$$= kI_f(uv). \qquad \text{Q.E.D.}$$

In the next theorem we use the notion of *betweenness* in a way which is meant not to exclude identity with one of the end points.

THEOREM 2. *If the points $f(x)$, $f(y)$ and $f(z)$ are collinear and $f(y)$ is between $f(x)$ and $f(z)$ then*

$$I_f(xy) + I_f(yz) = I_f(xz).$$

PROOF: Extending our subscript notation, let $f(x) = \mathbf{x}$, etc. Since the three points $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are collinear, and $\mathbf{y}$ is between $\mathbf{x}$ and $\mathbf{z}$, there is a number k such that $0 \leq k \leq 1$ and

(1) $$\mathbf{y} = k\mathbf{x} + (1 - k)\mathbf{z},$$

whence

$$\mathbf{y} - \mathbf{z} = k(\mathbf{x} - \mathbf{z}),$$

and thus by Theorem 1

(2) $$I_f(yz) = kI_f(xz).$$

By adding and subtracting $\mathbf{x}$ from the right-hand side of (1), we get:

$$\mathbf{y} = k\mathbf{x} + (1 - k)\mathbf{z} + \mathbf{x} - \mathbf{x},$$

whence

$$\mathbf{x} - \mathbf{y} = (1 - k)(\mathbf{x} - \mathbf{z}),$$

and thus by virtue of Theorem 1 again,

(3) $$I_f(xy) = (1 - k)I_f(xz).$$

Adding (2) and (3) we obtain the desired result:

$$I_f(xy) + I_f(yz) = I_f(xz). \qquad \text{Q.E.D.}$$

Our next objective is to prove a partial converse of Theorem 2. Since the notion of Lorentz transformation is needed in the proof, we introduce the appropriate formal definitions at this point. $\mathscr{I}$ is the identity matrix of the necessary order.

DEFINITION 4. *A matrix $\mathscr{A}$ (of order 4) is a* LORENTZ MATRIX *if and only if there exist real numbers β, δ, a three-dimensional vector U, and an orthogonal matrix $\mathscr{E}$ of order 3 such that*

$$\beta^2\left(1-\frac{U^2}{c^2}\right)=1$$

$$\delta^2 = 1$$

$$\mathscr{A}=\begin{pmatrix}\mathscr{E} & 0\\ 0 & \delta\end{pmatrix}\begin{pmatrix}\mathscr{I}+\dfrac{\beta-1}{U^2}\,U^*U & -\dfrac{\beta U^*}{c^2}\\ -\beta U & \beta\end{pmatrix}.$$

(In this definition and elsewhere, if A is a matrix, A^* is its transpose, and vectors like U are one-rowed matrices — thus U^* is a one-column matrix.) The physical interpretation of the various quantities in Definition 1 should be obvious. The number β is the *Lorentz contraction factor.* When $\delta = -1$, we have a reversal of the direction of time. The matrix $\mathscr{E}$ represents a *rotation* of the spatial coordinates, or a rotation followed by a reflection. The vector U is the *relative velocity* of the two frames of reference. For future reference it may be noted that every Lorentz matrix is non-singular.

DEFINITION 5. *A Lorentz transformation is a one-one function φ mapping R_4 onto itself such that there is a Lorentz matrix $\mathscr{A}$ and a 4-dimensional vector B so that for all Z in R_4*

$$\varphi(Z) = Z\mathscr{A} + B.$$

The physical interpretation of the vector B is clear. Its first three coordinates represent a translation of the origin of the spatial coordinates, and its last coordinate a translation of the time origin. Definition 5 makes it clear that every Lorentz transformation is a nonsingular affine transformation of R_4, a fact which we shall use in several contexts. The important consideration for the proof of Theorem 3 is that affine transformations preserve the collinearity of points.

THEOREM 3. *If any two of the three points x, y, z are distinct and lie on an inertial path with respect to f and if $I_f(xy) + I_f(yz) = I_f(xz)$, then the points $f(x)$, $f(y)$ and $f(z)$ are collinear, and $f(y)$ is between $f(x)$ and $f(z)$.*

PROOF: Three cases naturally arise.

Case 1. $I^2(xy) < 0$. In this case the line segment $f(x) - f(y)$ is an inertial path segment from x to y, and there exists a Lorentz transformation φ which will transform the segment $f(x) - f(y)$ to "rest", that is, more precisely, φ may be chosen so as to transform f to a frame f', which need not be a member of $\mathfrak{F}$, such that the spatial coordinates of x and y are at the origin, the time coordinate of x is zero, and z has but one spatial coordinate, by appropriate spatial rotation. That is, we have:

$$f'(x) = \langle 0, 0, 0, 0 \rangle,$$

$$f'(y) = \langle 0, 0, 0, y_4' \rangle,$$

$$f'(z) = \langle z_1', 0, 0, z_4' \rangle.$$

We shall prove that $f'(x)$, $f'(y)$ and $f'(z)$ are collinear. Since φ is non-singular and affine, its inverse φ^{-1} exists and is affine, whence collinearity is preserved in transforming from f' back to f.

It is a familiar fact that the relativistic intervals $I_f(xy)$, $I_f(yz)$ and $I_f(xz)$ are Lorentz invariant and thus have the same value with respect to f' as f. Consequently, from the additive hypothesis of the theorem, we have:

$$\sqrt{-c^2 y_4'^2} + \sqrt{z_1'^2 - c^2(y_4' - z_4')^2} = \sqrt{{}_1'^2 - c^2 z_4'^2}. \tag{1}$$

Squaring both sides of (1), then cancelling and rearranging terms, we obtain:

$$\sqrt{-y_4'^2} \cdot \sqrt{z_1'^2 - c^2(y_4' - z_4')^2} = c y_4'(y_4' - z_4'). \tag{2}$$

If $y_4' = 0$, then x and y are identical, contrary to the hypothesis that $I^2(xy) < 0$. Taking then $y_4' \neq 0$, dividing it out in (2), squaring both sides and cancelling, we infer:

$$-z_1'^2 = 0,$$

whence

$$z_1' = 0,$$

which establishes the collinearity in f' of the three points, since their spatial coordinates coincide, and obviously $f'(y)$ is between $f'(x)$ and $f'(z)$.

Case 2. $I_f^2(yz) < 0$. Proof similar to Case 1.

Case 3. $I_f^2(xz) < 0$. By an argument similar to that given for Case 1, we may go from f to a frame f' by a Lorentz transformation which will transform the inertial segment $f(x) - f(z)$ to "rest." That is, we obtain:

$$f'(x) = \langle 0, 0, 0, 0 \rangle,$$
$$f'(y) = \langle y_1', 0, 0, y_4' \rangle,$$
$$f'(z) = \langle 0, 0, 0, z_4' \rangle.$$

Then by the additive hypothesis of the theorem:

$$\sqrt{y_1'^2 - c^2 y_4'^2} + \sqrt{y_1'^2 - c^2(y_4' - z_4')^2} = \sqrt{-c^2 z_4'^2}. \tag{3}$$

Proceeding as before, by squaring and cancelling, we obtain from (3):

$$\sqrt{-c^2 z_4'^2} \cdot \sqrt{y_1'^2 - c^2 y_4'^2} = -c^2 y_4' z_4'. \tag{4}$$

Squaring again and cancelling yields:

$$y_1'^2 z_4'^2 = 0. \tag{5}$$

There are now two possibilities to consider: either $y_1' = 0$ or $z_4' = 0$. If the former is the case, then the three points are collinear in R_4, for they are all three placed at the origin of the spatial coordinates. On the other hand, if $z_4' = 0$, then x and z are identical points, contrary to hypothesis. Again it is obvious that $f'(y)$ is between $f'(x)$ and $f'(z)$. Q.E.D.

That a full converse of Theorem 2 cannot be proved, in other words that the additive hypothesis

$$I_f(xy) + I_f(yz) = I_f(xz)$$

does not imply collinearity, is shown by the following counterexample:

$$f(x) = \langle 0, 0, 0, 0 \rangle,$$
$$f(y) = \langle 1, 1, 0, 0 \rangle$$
$$f(z) = \langle \sqrt{2}c, 0, 0, 1 \rangle.$$

Clearly, $f(x)$, $f(y)$ and $f(z)$ are not collinear in R_4, but $I_f(xy) + I_f(yz) = I_f(xz)$, that is,

$$\sqrt{2} + \sqrt{(1 - \sqrt{2}c)^2 + 1 - c^2} = \sqrt{2c^2 - c^2}. \tag{1}$$

For, simplifying and rearranging (1), we see it is equivalent to:

$$\sqrt{2 - 2\sqrt{2}c + c^2} = c - \sqrt{2} \tag{2}$$

and the left-hand of (2) is simply

$$\sqrt{(c - \sqrt{2})^2} = c - \sqrt{2}.$$

(It may be mentioned that the full converse of Theorem 2 does hold for R_2, that is, when there is a restriction to one spatial dimension.)

We now want to prove some theorems about properties which are invariant in $\mathfrak{F}$. Formally, a property is *invariant* in $\mathfrak{F}$ if and only if it holds or does not hold uniformly for every member f of $\mathfrak{F}$. Thus to say that the property of a line being an inertial path is invariant in $\mathfrak{F}$ means that a line with respect to f in $\mathfrak{F}$, is an inertial path with respect to f if and only if it is an inertial path with respect to every f' in $\mathfrak{F}$. All geometric objects referred to here are with respect to the frames in $\mathfrak{F}$.

THEOREM 4. *The property of being the midpoint of a finite segment of an inertial path is invariant in* $\mathfrak{F}$.

PROOF: Suppose x, y and z lie on an inertial path with respect to f and

(1) $$f(y) = \tfrac{1}{2}f(x) + \tfrac{1}{2}f(z),$$

and thus

$$f(y) - f(x) = \tfrac{1}{2}[f(z) - f(x)].$$

Consequently by virtue of Theorem 1

(2) $$I_f(xy) = \tfrac{1}{2}I_f(xz)$$

and similarly

(3) $$I_f(yz) = \tfrac{1}{2}I_f(xz),$$

whence

(4) $$I_f(xy) + I_f(yz) = I_f(xz).$$

Now by the invariance axiom of Definition 3, for any f' in $\mathfrak{F}$

$$I_{f'}(xy) = I_f(xy)$$
$$I_{f'}(yz) = I_f(yz)$$
$$I_{f'}(xz) = I_f(xz).$$

Substituting these identities in (4) we obtain:

$$I_{f'}(xy) + I_{f'}(yz) = I_{f'}(xz).$$

Thus by virtue of Theorem 3, $f'(x)$, $f'(y)$ and $f'(z)$ are collinear with $f'(y)$ between $f'(x)$ and $f'(z)$. Moreover, since by the invariance axiom (2) and (3) hold for f', we conclude $f'(y)$ is actually the midpoint. Q.E.D.

This proof is easily extended to show that the property of being an inertial path is invariant in $\mathfrak{F}$, but we do not directly need this fact. We next want to show that this midpoint property is invariant for arbitrary segments. In view of the counterexample following Theorem 3 it is evident that a direct proof in terms of the relativistic intervals cannot be given. The method we shall use consists essentially of constructing a parallelogram whose sides are segments of inertial paths. A similar but somewhat more complicated proof is given in Rubin and Suppes [3].

THEOREM 5. *The property of being the midpoint of an arbitrary finite segment is invariant in* $\mathfrak{F}$.

PROOF: Let $A = \langle Z_1, t_1\rangle$ and $B = \langle Z_2, t_2\rangle$ where A is an arbitrary segment in R_4. (The points A to G defined here are with respect to f in $\mathfrak{F}$.) For definiteness assume $t_1 \geq t_2$. We set

$$Z_0 = \frac{Z_1 + Z_2}{2}$$

and we choose t_0 and t_3 so that

$$t_0 < t_2 - \frac{|Z_1 - Z_2|}{2c},$$

$$t_3 > t_1 + \frac{|Z_1 - Z_2|}{2c},$$

$$|A - \langle Z_0, t_3\rangle| = |\langle Z_0, t_0\rangle - B|,$$

$$|A - \langle Z_0, t_0\rangle| = |\langle Z_0, t_3\rangle - B|.$$

We now let (see Figure 1)

$$C = \langle Z_0, t_0\rangle, \qquad D = \langle Z_0, t_3\rangle,$$

$$E = \frac{A + B}{2}, \qquad F = \frac{B + D}{2},$$

$$G = \frac{A + C}{2}.$$

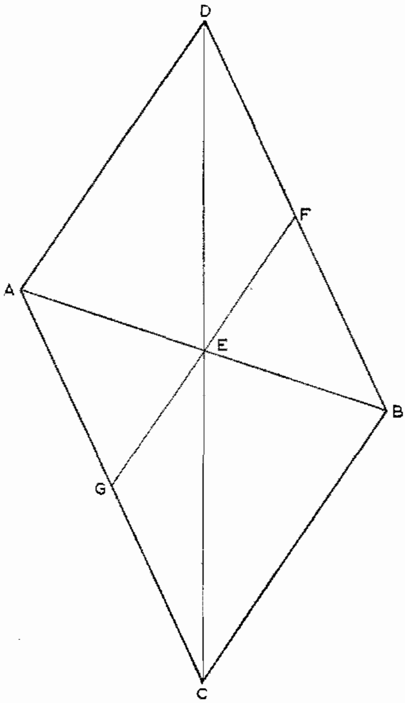

Fig. 1

Denoting now the same points with respect to f' in $\mathfrak{F}$ by primes, we have by virtue of this construction in f and the invariance property of Theorem 4,

(1) $E' = \frac{1}{2}(C' + D')$,

(2) $F' = \frac{1}{2}(B' + D')$,

(3) $G' = \frac{1}{2}(A' + C')$,

(4) $E' = \frac{1}{2}(F' + G')$.

Substituting (2) and (3) into (4) we have:

$$E' = \tfrac{1}{2}[\tfrac{1}{2}(B' + D') + \tfrac{1}{2}(A' + C')]$$
$$= \tfrac{1}{2}[\tfrac{1}{2}(A' + B') + \tfrac{1}{2}(C' + D')].$$

Now substituting (1) into the right-hand side of the last equation and simplifying, we infer the desired result:

$$E' = \tfrac{1}{2}(A' + B'),$$

since by construction $E = \frac{1}{2}(A + B)$. Thus the midpoint of an arbitrary segment is preserved. Q.E.D.

THEOREM 6. *The property of two finite segments of inertial paths being parallel and in a fixed ratio is invariant in* $\mathfrak{F}$.

PROOF: Let $f(x) - f(y) = k[f(u) - f(v)]$, with $f(x) - f(y)$ and $f(u) - f(v)$ segments of inertial paths. Without loss of generality we may assume $k \geq 1$. Let z be the point such that $f(x) - f(y) = k[f(x) - f(z)]$. We now construct a parallelogram with $f(u) - f(v)$ and $f(x) - f(z)$ as two parallel sides. By the previous theorem any parallelogram in f is carried into a parallelogram in f' since the midpoint of the diagonals is preserved. Thus

(1) $$f'(u) - f'(v) = f'(x) - f'(z),$$

but by Theorems 2 and 3

(2) $$f'(x) - f'(y) = k[f'(x) - f'(z)],$$

(for details see proof of Theorem 4), whence from (1) and (2)

$$f'(x) - f'(y) = k[f'(u) - f'(v)]. \qquad \text{Q.E.D.}$$

As the final theorem about properties invariant in $\mathfrak{F}$, we want to generalize the preceding theorem to arbitrary finite segments.

THEOREM 7. *The property of two arbitrary finite segments being parallel and in a fixed ratio is invariant in* $\mathfrak{F}$.

PROOF: In view of preceding theorems, the crucial thing to show is that if

$$f(x) - f(y) = k[f(x) - f(z)]$$

then

$$f'(x) - f'(y) = k[f'(x) - f'(z)].$$

Our approach is to use an "inertial" parallelogram similar to the one used in the proof of Theorem 5. In fact an exactly similar construction will be used; points A to E are constructed identically, where $A = f(x)$ and $B = f(y)$. Without loss of generality we may assume $k > 2$, that is, that $f(z) = F$ is between A and E. We then have that

(1) $$A - E = \frac{k}{2}[A - F].$$

We draw through F a line parallel to CD, which cuts AC at G and AD at H. (See Figure 2.)
Now (1) is equivalent to:

(2) $$F = \left(1 - \frac{2}{k}\right)A + \frac{2}{k}E.$$

Moreover, by construction

(3) $$F = \tfrac{1}{2}(G + H)$$

(4) $$E = \tfrac{1}{2}(C + D)$$

(5) $$G = \left(1 - \frac{2}{k}\right)A + \frac{2}{k}C$$

(6) $$H = \left(1 - \frac{2}{k}\right)A + \frac{2}{k}D.$$

Since GFH, AGC, AHD and CED are by construction segments of in-

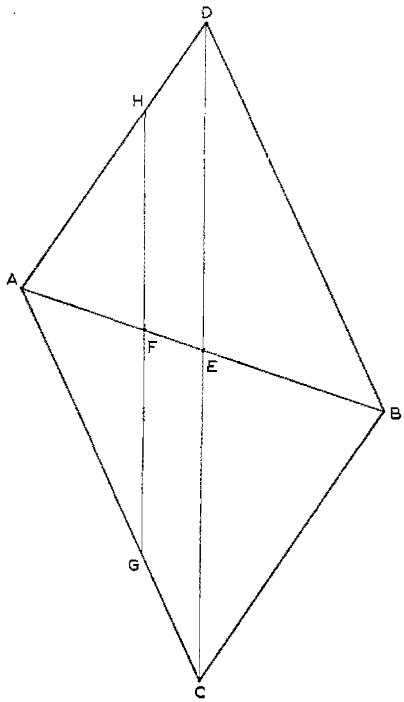

Fig. 2

ertial paths, by virtue of Theorem 7, we have from (3)–(6):

(7) $F' = \frac{1}{2}(G' + H')$

(8) $E' = \frac{1}{2}(C' + D')$

(9) $G' = \left(1 - \frac{2}{k}\right)A' + \frac{2}{k}C'$

(10) $H' = \left(1 - \frac{2}{k}\right)A' + \frac{2}{k}D'.$

Substituting (9) and (10) in (7), we get:

(11) $$F' = \left(1 - \frac{2}{k}\right)A' + \frac{1}{k}(C' + D').$$

And now substituting (8) in (11), we obtain the desired result:

(12) $F' = \left(1 - \frac{2}{k}\right)A' + \frac{2}{k}E'.$

But now by virtue of Theorem 5

$$E' = \tfrac{1}{2}(A' + B'),$$

which together with (12) yields:

$$F' = \left(1 - \frac{1}{k}\right)A' + \frac{1}{k}B',$$

which is equivalent to:

(13) $$f'(x) - f'(y) = k[f'(x) - f'(z)].$$

The remainder of the proof, based upon considering $f(x) - f(y) = k[f(u) - f(v)]$, is exactly like that of Theorem 6 and may be omitted. (In place of Theorems 2 and 3 in that proof we use the result just established.)

Q.E.D.

We now state the theorem toward which the preceding seven have been directed.

THEOREM 8. *Any two frames in $\mathfrak{F}$ are related by a non-singular affine transformation.*

PROOF: A familiar necessary and sufficient condition that a transformation of a vector space be affine is that parallel finite segments with a fixed ratio be carried into parallel segments with the same fixed ratio. (See, e.g. Birkhoff and MacLane [1, p. 263].) Hence by virtue of Theorem 7 any two frames are related by an affine transformation. Non-singularity of the transformation follows from the fact that each frame in $\mathfrak{F}$ is a one-one mapping of X onto R_4. Q.E.D.

Once we have any two frames in $\mathfrak{F}$ related by an affine transformation, it is not difficult to proceed to show that they are related by a Lorentz transformation. In the proof of this latter fact, it is convenient to use a Lemma about Lorentz matrices, which is proved in Rubin and Suppes [3], and is simply a matter of direct computation.

LEMMA 1. *A matrix $\mathscr{A}$ (of order 4) is a Lorentz matrix if and only if*

$$\mathscr{A}\begin{pmatrix}\mathscr{I} & 0\\ 0 & -c^2\end{pmatrix}\mathscr{A}^* = \begin{pmatrix}\mathscr{I} & 0\\ 0 & -c^2\end{pmatrix}$$

We now prove the basic result:

THEOREM 9. *Any two frames in $\mathfrak{F}$ are related by a Lorentz transformation.*

PROOF: Let f, f' be two frames in $\mathfrak{F}$. As before, for x in X, $f(x) = \boldsymbol{x}$, $f_1(x) = x_1$, $f'(x) = \boldsymbol{x}'$, etc. We consider the transformation φ such that for every x in X, $\varphi(\boldsymbol{x}) = \boldsymbol{x}'$. By virtue of Theorem 8 there is a non-singular matrix (of order 4) and a four-dimensional vector B such that for every x in X

$$\varphi(\boldsymbol{x}) = \boldsymbol{x}\mathscr{A} + B.$$

The proof reduces to showing that $\mathscr{A}$ is a Lorentz matrix.

Let

$$\mathscr{A} = \begin{pmatrix}\mathscr{D} & E^*\\ F & g\end{pmatrix}. \tag{1}$$

And let α be a light line (in f) such that for any two distinct points $\boldsymbol{x}$ and $\boldsymbol{y}$

of α if $\boldsymbol{x} = \langle Z_1, t_1 \rangle$ and $\boldsymbol{y} = \langle Z_2, t_2 \rangle$, then

$$\frac{Z_1 - Z_2}{t_1 - t_2} = W. \tag{2}$$

Clearly $|W| = c$. Now let

$$W' = \frac{Z_1' - Z_2'}{t_1' - t_2'}. \tag{3}$$

From (1), (2) and (3) we have:

$$W' = \frac{(Z_1 - Z_2)\mathscr{D} + (t_1 - t_2)F}{(Z_1 - Z_2)E^* + (t_1 - t_2)g} \tag{4}$$

Dividing all terms on the right of (4) by $t_1 - t_2$, and using (2), we obtain:

$$W' = \frac{W\mathscr{D} + F}{WE^* + g}. \tag{5}$$

At this point in the argument we need to know that $|W'| = c$, that is to say, we need to know that if $I_f(xy) = 0$, then $I_{f'}(xy) = 0$. The proof of this fact is not difficult. From our fundamental invariance axiom we have that $I_{f'}(xy) \geq 0$, that is,

$$|W'| \geq c. \tag{6}$$

Consider now a sequence of inertial lines $\alpha_1, \alpha_2, \ldots$ whose slopes $W_1, W_2, \ldots$ are such that

$$\lim_{n \to \infty} W_n = W. \tag{7}$$

Now corresponding to (5) we have:

$$|W_n'| = \left| \frac{W_n\mathscr{D} + F}{W_nE^* + g} \right| < c. \tag{8}$$

Whence, from (8) we conclude that if $WE^* + g \neq 0$, then

$$|W'| = |\lim_{n \to \infty} W_n'| \leq c. \tag{9}$$

Thus from (6) and (9) we infer

$$|W'| = c, \tag{10}$$

if $WE^* + g \neq 0$, but that this is so is easily seen. For, suppose not. Then

$$\lim_{n\to\infty}(W_nE^* + g) = 0,$$

and thus

$$\lim_{n\to\infty}(W_n\mathscr{D} + F) = 0.$$

Consequently $W\mathscr{D} + F = 0$, and $\langle W, 1\rangle\mathscr{A} = 0$, which is absurd in view of the non-singularity of $\mathscr{A}$.

Since $|W'| = c$, we have by squaring (5):

$$\frac{W\mathscr{D}\mathscr{D}^*W^* + 2W\mathscr{D}F^* + |F|^2}{(WE^* + g)^2} = c^2, \tag{11}$$

and consequently

$$W(\mathscr{D}\mathscr{D}^* - c^2E^*E)W^* + 2W(\mathscr{D}F^* - c^2E^*g) + |F|^2 - c^2g = 0. \tag{12}$$

Since (12) holds for an arbitrary light line, we may replace W by $- W$, and obtain (12) again. We thus infer:

$$W(\mathscr{D}F^* - c^2E^*g) = 0,$$

but the direction of W is arbitrary, whence

$$\mathscr{D}F^* - c^2E^*g = 0. \tag{13}$$

Now let $\boldsymbol{x} = \langle 0, 0, 0, 0\rangle$ and $\boldsymbol{y} = \langle 0, 0, 0, 1\rangle$. Then

$$I_f^2(xy) = - c^2.$$

But it is easily seen from (1) that

$$I_{f'}^2(xy) = |F|^2 - c^2g^2,$$

and thus by our fundamental invariance axiom

$$c^2g^2 - |F|^2 = c^2. \tag{14}$$

From (12), (13), (14) and the fact that $|W|^2 = c^2$, we infer:

$$W(\mathscr{D}\mathscr{D}^* - c^2E^*E)W^* = |W|^2,$$

and because the direction of W is arbitrary we conclude:

$$\mathscr{D}\mathscr{D}^* - c^2E^*E = \mathscr{I}, \tag{15}$$

where $\mathscr{I}$ is the identity matrix.

Now by direct computation on the basis of (1),

$$(16)\qquad \mathscr{A}\begin{pmatrix}\mathscr{I} & 0\\ 0 & -c^2\end{pmatrix}\mathscr{A}^* = \begin{pmatrix}\mathscr{D}\mathscr{D}^* - c^2E^*E & \mathscr{D}F^* - c^2E^*g\\ (\mathscr{D}F^* - c^2E^*g)^* & FF^* - c^2g^2\end{pmatrix}$$

From (13), (14), (15) and (16) we arrive finally at the result:

$$\mathscr{A}\begin{pmatrix}\mathscr{I} & 0\\ 0 & -c^2\end{pmatrix}\mathscr{A}^* = \begin{pmatrix}\mathscr{I} & 0\\ 0 & -c^2\end{pmatrix},$$

and thus by virtue of Lemma 1, $\mathscr{A}$ is a Lorentz matrix. Q.E.D.

4. **Temporal Parity.** Turning now to problems of parity, we may for simplicity restrict the discussion to time reversals. Similar considerations apply to spatial reflections.

A simple axiom, which will prevent time reversal between frames in $\mathfrak{F}$, is:

(T1) *There are elements x and y in X such that for all f in* $\mathfrak{F}$

$$f_4(x) < f_4(y).$$

There is, however, a simple objection to this axiom. It is unsatisfactory to have time reversal depend on the existence of special space-time points, which could possibly occur only in some remote region or epoch. This objection is met by T2.

(T2) *If* $I_f^2(xy) < 0$ *then either for all f in* $\mathfrak{F}$

$$f_4(x) < f_4(y)$$

or for all f in $\mathfrak{F}$

$$f_4(y) < f_4(x).$$

T2 replaces the postulation of special points by a general property: given any segment of an inertial path, all frames in $\mathfrak{F}$ must orient the direction of time for this segment in the same way.

Nevertheless, there is another objection to T1 which holds also for T2: the appropriate axiom should be formulated so that a given observer in a frame f may verify it without observing any other frames, that is, he may decide if he is a qualified candidate for membership in $\mathfrak{F}$ without observing other members of $\mathfrak{F}$. (This issue is relevant to the single axiom of Definition 3 but cannot be entered into here.) From a logical standpoint this means eliminating quantification over elements of $\mathfrak{F}$, which may be done by introducing a fourth primitive notion, a binary relation σ of *signaling* on X. To block time reversal we need postulate but two properties of σ:

(T3.1) *For every x in X there is a y in X such that $x\sigma y$.*

(T3.2) *If $x\sigma y$ then $f_4(x) < f_4(y)$.*

However, a third objection to (T1) also applies to (T2) and (T3). Namely, we are essentially postulating what we want to prove. The axioms stated here correspond to postulating artifically in a theory of measurement of mass that a certain object must be assigned the mass of one. I pose the question: *Is it possible to find "natural" axioms which fix a direction of time?* It may be mentioned that Robb's meticulous axiomatization [2] in terms of the notion of *after* provides no answer.

References

[1] BIRKHOFF, G. and S. MACLANE, *A Survey of Modern Algebra*. New York 1941, XI+450 pp.

[2] ROBB, A. A., *Geometry of Space and Time*. Cambridge 1936, VII + 408 pp.

[3] RUBIN, H. and P. SUPPES, *Transformations of systems of relativistic particle mechanics*. Pacific Journal of Mathematics, vol. 4 (1954), pp. 563–601.