H.G. Natke & Y. Ben-Haim (Eds.), Uncertainty: Models and Measurement, Essays in Honor of R. Duncan Luce. Mahwah, NJ: Lawrence Erlbaum, 1997, pp. 103-109.

FREEDOM AND UNCERTAINTY

Patrick Suppes

There are many issues about freedom to concentrate on. In this paper, I concern myself with three of the significant topics, the last two of which have not been much discussed in the literature on freedom. In the first section, I review the classical concept of freedom as absence from constraints. In the second section, I analyze the role of uncertainty in relation to freedom, and in the final section I propose an analysis of the relation between learning and freedom.

1 Freedom as Absence of Constraints

There is little doubt that both the common-sense concept of freedom and many philosophical analyses of freedom concentrate on the characterization of freedom as the absence of constraints. Moreover, such constraints are almost immediately qualified. It is not ordinarily considered a constraint on freedom that as agents our bodies must obey the laws of physics. So, the fact that we cannot freely jump as high as we please is not regarded as a relevant constraint on freedom, because it is not a constraint by another agent but a constraint that must satisfy the laws of physics, in particular, the law of gravitation.

A familiar qualification of the constraints relevant to freedom are those imposed by another agent. A free action of an agent is often characterized as one that is not compelled or directed by another agent. There is much that has been said about this notion of freedom as absence of constraints by other agents. Here, however, I shall only discuss rather briefly a few central topics.

A first issue is the problem of internal psychological constraints. Both in folk psychology and in the law, it is common to say that an individual did not freely commit a certain crime to which he has confessed, and therefore, is not guilty. The reason given is that the individual was subject to overwhelmingly strong irrational compulsions. In some cases, a plea of insanity is upheld.

There is certainly often a question of uncertainty about such judgments, but in this kind of context, the uncertainty is about the evaluation of the true state of mind of the person who admittedly committed the crime. This is uncertainty about an evidential claim and is not the kind of uncertainty I focus on in the next section. Uncertainty about the correctness of a psychological claim concerning the state of mind of an individual is not in any direct sense a claim that uncertainty is intrinsic to freedom, which is my central topic, and so I pursue such psychological questions about internal compulsions no further here.

Another familiar argument is that freedom can be a proper part of folk psychology, but at a deeper level, the very idea of freedom is an illusion, because everything is causally determined. Perhaps the most famous philosopher to advocate these two positions together, that is, the one of freedom as the absence of constraints by other agents and the doctrine of causal determinacy, was Hume. Hume's agenda is in certain respects rather special. In his famous chapter on liberty and necessity in Part III, Book Two of *A Treatise of Human Nature* (1739/1888), Hume wants to make the case for there being a science of the mind comparable to the science of physics and, more generally of natural science, exemplified by the recent triumphs in physics, especially those of Newton. He readily admits that we cannot give a detailed explanation, from a scientific standpoint, of much mental phenomena, but he rightly makes the point this is also true of physical phenomena. So he makes the claim that there is just as much reason to believe in necessity in the case of mental phenomena as in the case of physical phenomena. His point is to deny any absolute concept of liberty or freedom. Everything, physical or mental, is causally determined, as we would formulate the concept today or, as he would put it, causally necessary.

Another great philosopher who held similar views was Immanuel Kant. Within the realm of experience, Kant had a variety of detailed arguments as to why we should view all phenomena in experience as governed by the laws of nature. by which, he meant the laws of physics considered in a broad way. In the Third Causal Antinomy in the *Critique of Pure Reason* (1781/1787), Kant asserts as the Thesis of the antinomy that the idea of a determinant sequence of causes extending ever backward in time is absurd. Any causal sequence must begin with an event that is absolutely spontaneous (freedom in nature) and is the first member of the sequence. He rejects. however, this argument in the Antithesis and accepts throughout as part of his philosophical doctrine the complete determinism, or, as he (and Hume) would say, the necessity of the laws of nature. There is great subtlety about Kant's argument. A case can certainly be maintained that his final decision in analyzing the antinomies, in particular the Third Antinomy, was to make the concept of determinate causation a regulative idea and to admit that a completely compelling argument for its constitutive character could not be given.

Kant is also famous for having two other concepts of freedom. First is the concept of transcendental freedom, which is outside experience, that is, outside the framework of time and space, and therefore outside the laws of physics. The other concept is that of practical freedom, which is in many respects in its philosophical roots like Hume's concept of freedom as absence from constraints. However, I hasten to add there is much more to Kant's concept of practical freedom and it is very much entwined with his concept of morality.

Still another issue for the agent-constraint view of freedom is that of the extent to which other animals possess such freedom. There is certainly a long tradition, related to both moral and theological concepts, that admits no place for freedom in the behavior of animals, but to many of us, this seems rather ridiculous from the standpoint of modern biological ideas of evolution. There remain, however, even within the biological framework, issues about freedom for animals, especially as we go down the phylogenetic scale. Do aplysia have freedom? As much as I would like to pursue further arguments here, all I want to say at this point is that the conception of freedom as absence from constraint by other agents has something important and correct about it. It does not mean that it is a complete and satisfactory analysis in all respects.

2 Uncertainty as Essential[1]

The close connection between freedom and uncertainty is the main focus of this paper. Entropy as the measurement of freedom is also a focus of this section. The deeper reasons, derived from ergodic theory, for using this particular measure of uncertainty are developed later. The central idea is that two elections or markets as processes have the same freedom if their uncertainty structures are isomorphic. The technical details are given in what follows, but what is to be emphasized to begin with is that even the suggestion that uncertainty is central to the fact of freedom is missing in the classical philosophical analyses mentioned above, and in the main philosophical successors to Hume and Kant, such as John Stuart Mill in his famous essay *On Liberty* (1859/1991). This omission continues in the standard literature of this century. Throughout the rest of this paper I try to show that this omission is mistaken, and that intuitive features of freedom in many economic, political and social settings implicitly take some form of uncertainty for granted.

To put the focus on uncertainty, I propose entropy as *the* measurement of freedom. Entropy is already used as a measure of uncertainty in mathematical statistics and statistical mechanics. Other features of freedom may also be subject to measurement, but my claim is that uncertainty, which is particularly susceptible of measurement, is, as a measure of freedom, *primus inter pares.*

Entropy as a proposed measurement of freedom is phenomenological and result, rather than procedurally, oriented. Consider two elections. The first, E_1, has three candidates and each receives about 1/3 of the votes. The second, E_2, has two candidates and the winner of the two receives about 3/4 of the votes. Almost all of us would agree, I think, that the results as such are evidence of E_1 being more free than E_2. In saying this we are assuming the usual *ceteris paribus* conditions. Moreover, in matters political or economic there is a strong skeptical tradition that looks to results rather than intentions in judging the character of an institution or procedure.

I propose that we measure the freedom of a set A of alternatives by the entropy H of the actual chosen proportions, or relative frequencies, of the various alternatives, that is,

$$H(A) = -\sum_{i \varepsilon A} p_i \log p_i,$$

where log is to the base 2, $p_i \geq 0$ and if $p_i = 0$ then $0 \log 0 = 0$.

To give a feeling for the numbers, so to speak, let us consider the entropy of some American presidential elections. According to the measure proposed, the elections since 1850 with the maximum freedom, i.e., maximum entropy, of popular vote were

[1] Much of the content of this section is taken from my recent article, Suppes (1996).

those of 1860 and 1912. The tallies were as follows[2]:

1860		1912	
Abraham Lincoln	1,865,593	Woodrow Wilson	6,296,547
J. C. Breckinridge	848,356	Theodore Roosevelt	4,118,571
Stephen A. Douglas	1,382,713	William H. Taft	3,486,720
John Bell	592,906	Eugene V. Debs	900.672
		Eugene W. Chafin	206,275
		Arthur E. Reimer	28,750
	H=1.87		H=1.87

In contrast, the least free election as measured by popular vote was in 1964, with an entropy measure of 0.98.

	1964		
Lyndon B. Johnson	43,129,566	Clifton DeBerry	32,720
Barry M. Goldwater	27,178,188	E. Harold Muun	23.267
Eric Hass	45,219		

The measure of freedom I am proposing is, as I said at the beginning, mainly phenomenological. There is no suggestion that the measure itself says very much about the causal factors producing the measure at a given time, or a change in the measure from one period to another, whether in an election or in a market. There is, surely, an utter pluralism of causes of changes in entropy. Above all, increases in freedom occur not necessarily because of the intentional actions of individuals focusing on problems of freedom, but often because of what Aristotle termed incidental causation. This means that their intentions were focused on something else, but out of those intentions arose a mixture of results from the actions of many individuals that increased or decreased the freedom of a given institution, or political or social procedure.

It may well be said by some political philosophers, but not by politicians or voters, that we do not really care about the outcomes of elections. What we care about are the political conditions under which they take place. If there is good evidence prior to the elections that there was a serious campaign among alternative candidates and individuals could freely state their political opinions, then what we judge as important are these conditions and not the fact that there was a real landslide of 90% in the actual voting. In this sense, it would be argued, the entropy measure is inappropriate. There is something in this criticism. It means that the analysis of freedom should be displaced from the results of the election to the procedures or processes leading up to it. We should then attempt to measure the presence of genuine dissent in the political dialogue preceding the election, the opportunities for choosing in terms of external social and political pressures, the resources available to the various candidates, etc. In

[2]Data taken from *Historical Statistics of the United States, Colonial Times to 1970, Bicentennial Edition, Part II.* U.S. Bureau of the Census, 1975

my own view the outcome of this investigation would be in most cases fairly consistent with the analysis of the election results. Moreover, it is difficult to get quantitative and objective data about much of the political process leading to elections, but assuming the elections are themselves not dishonestly run, excellent quantitative data can be found in the results alone.

When there is freedom in the sense of entropy as measured quantitatively and as proposed here, it would be surprising to have a high measure of freedom for the process and a low one for the result. Notice, of course, that it is part of the rhetoric of politics that many people would say, even when very few resources were available, that it is still the case that individuals under the law were free to speak their minds about the candidates and to campaign as they wished in favor of whomever they wished. This is an important aspect of freedom and one that may not be satisfactorily caught by the measure I am proposing, but it is also one that is a source of skepticism about a political process that permits the kind of freedom just described and yet produces almost no results to back it up.

Stochastic Freedom. There is another sense of process that is central to the view of freedom being developed. We can observe successive elections and markets for a number of time periods. It is, above all, the entropy rate of these processes over time, rather than data for a single cross-section, that is central, for reasons I hope to make clear.

First, some technical details. A stochastic process $\mathcal{X}$ is an indexed family $\{\mathbf{X_n}\}$ of random variables. The index, discrete or continuous, is usually interpreted as time, and so it will be here. For simplicity and without any real conceptual loss, I consider only the discrete case with $n = 1, 2, 3, \ldots$, although some remarks will concern the doubly infinite case, $n = \ldots -2, -1, 0, 1, 2, \ldots$. The usual assumption about the collection of joint probability distributions of any finite subsequence of the random variables being consistent is made.

The appropriate concept of entropy for a stochastic process $\mathcal{X}$ is that of *entropy rate* $H(\mathcal{X})$ defined as follows

$$H(\mathcal{X}) = \lim_{n\to\infty} \frac{1}{n} H(\mathbf{X_1}, \ldots, \mathbf{X_n}),$$

provided the limit exists. (Notice that $H(\mathbf{X_1}, \ldots, \mathbf{X_n})$ is just the entropy of the first n random variables. We convert to a rate by dividing by n.)

A (discrete, finite) Bernoulli process is a stochastic process that is a sequence $\mathbf{X_1}, \mathbf{X_2}, \ldots$, or possibly a doubly infinite sequence, with the $\mathbf{X_n}$'s independent and identically distributed random variables with a fixed finite range of values It is easy to show that such a Bernoulli process $\mathcal{X}$ has entropy rate

$$\begin{aligned} H(\mathcal{X}) &= \lim_{n\to\infty} \frac{H(\mathbf{X_1}, \mathbf{X_2}, \ldots, \mathbf{X_n})}{\mathbf{n}} = \frac{nH(\mathbf{X_1})}{\mathbf{n}} = H(\mathbf{X_1}) \\ &= -\sum p_i \log p_i . \end{aligned}$$

We take the measure of freedom to be the entropy rate of the process.

Consider a market over time in which m individuals are sellers and n are buyers. At each period each buyer makes a purchase from exactly one seller. As before, the uniform probability distribution on the set of m^n possible transactions would define a discrete (and finite-valued) Bernoulli process, which would be for m^n possible transactions the stochastic process with maximum entropy rate and thus the one of this size with maximum freedom.

I simplify the analysis at this point by considering only the sellers as the states of the market process. The probability of each of the m states, i.e., sellers, represents the probability a random buyer will choose that seller at the given time. In application of these ideas to market data we would often need to estimate $p_{i,n}$ for seller i at the end of time period n by the relative proportion of the market seller i had for that period and make no attempt to identify the behavior of individual buyers. This asymmetry in the treatment of buyers and sellers is common in the analysis of markets and correspondingly, in the case of elections for candidates and voters. However, it is to be emphasized that this limited kind of data analysis is not at all satisfactory for a study of market processes over time, when the entropy rate depends on the transition data for individual buyers, as will become clear in the sequel. I note here that a sample path for a buyer is the sequence of states occupied by the buyer from one time period to another, with the state representing the seller with whom the buyer has a transaction. Although I do not do it here, for actual data analysis it would be desirable to introduce a state corresponding to a buyer not making a transaction in a given time period. There is little doubt that most sellers would shudder at the utter randomness of a Bernoulli market from one period to the next, as would most candidates at a sequence of elections with a corresponding Bernoulli character. Many firms would accept, even if not maximally satisfied, a market that is about evenly divided among a relative small number of sellers, but would be aghast at the utter lack of customer loyalty as the buyers randomly shifted at each period from one seller to another.

The necessity of considering the time course of a market, and not just cross-section data, in measuring freedom can be well illustrated by a market with just three sellers. We can look at the three-state Markov market with the transition matrix

	1	2	3
1	$1-2\epsilon$	ϵ	ϵ
2	ϵ	$1-2\epsilon$	ϵ
3	ϵ	ϵ	$1-2\epsilon$

As $\epsilon \longrightarrow 0$, the entropy approaches zero, but the cross-sectional distribution remains $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. I think it is intuitively obvious that a market or election with 100% loyalty, i.e., with $\epsilon = 0$ in the above analysis, is not free. Sellers or candidates need make no effort to compete. This is why merely cross-section data can be misleading

More generally, for a stationary process the entropy rate as defined above, it can be shown, is equal to the conditional entropy rate, defined as

$$H'(\mathcal{X}) = \lim_{n \to \infty} H(\mathbf{X_n} | \mathbf{X_{n-1}}, \ldots, \mathbf{X_1}),$$

provided the limit exists, which it does for stationary processes. For a (first-order) stationary Markov process, as in our example,

$$\begin{aligned} H'(\mathcal{X}) &= \lim H(\mathbf{X_n} | \mathbf{X_{n-1}}, \ldots, \mathbf{X_1}) \\ &= H(\mathbf{X_2} | \mathbf{X_1}) \\ &= -\sum_x p(x) \sum_y p(y|x) \log p(y|x) \end{aligned}$$

and so it is easy to show for our Markov market example as defined above that as $\epsilon - 0$, $H(\mathcal{X}) \to 0$. (Hereafter, I drop the distinction between H and H' in view of their equality for stationary processes.)

I now turn to the concept that is critical for making entropy rate the essential measure of the freedom of a market or election process–I add the word "process" to emphasize we are considering processes, not one-time cross-sections. The central question is this. How do two markets, or a market and an election, for that matter, compare in their intuitive sense of freedom if they have the same entropy, and contrariwise? As far as I know, this is not a question that has been previously addressed in economics or political science. There have been several prior uses of entropy to measure the one-time cross-section distribution of a market, as part of a more general consideration of indices of concentration (Encaoua and Jacquemin, 1980, Curry and George, 1983, Tirole, 1988, Ch. 5, Foley, 1994), but not of a market as a stochastic process. More importantly, entropy, as an invariant feature of certain structural properties of stationary stochastic markets, has not been examined. The answer lies ready at hand in the mathematical literature on ergodic theory. In many cases of conceptual interest two stationary stochastic markets or elections will have the same entropy rate if and only if they are isomorphic in the measure-theoretic sense. It is this latter concept that needs to be formally defined.

Let us first begin with a standard probability space $(\Omega, \Im, P)$, where it is understood that $\Im$ is a σ-algebra of subsets of Ω and P is a σ-additive probability measure on $\Im$. We now consider a mapping T from Ω to Ω. We say that T is *measurable* if and only if whenever $A \in \Im$ then $T^{-1}A = \{\omega : T\omega \in A\} \in \Im$, and even more important, T is *measure preserving* if and only if $P(T^{-1}A) = P(A)$. T is *invertible* if the following three conditions hold: (i) T is $1-1$, (ii) $T\Omega = \Omega$, and (iii) if $A \in \Im$ then $TA = \{T\omega : \omega \in A\} \in \Im$. In the application we are interested in, each ω in Ω is a doubly infinite sequence and T is the *right-shift* such that if for all n, $\omega_n = \omega'_{n+1}$ then $T(\omega) = \omega'$. Intuitively this property corresponds to stationarity of the process—a time shift does not affect the probability laws of the process, and we can then use T to describe orbits or sample paths in Ω.

We now characterize isomorphism of two probability spaces on each of which there is given a measure-preserving transformation, whose domain and range need only be subsets of measure one, to avoid uninteresting complications with sets of measure zero that are subsets of Ω or Ω'. Thus we say $(\Omega, \Im, P, T)$ is *isomorphic in the measure-theoretic sense* to $(\Omega', \Im', P', T')$ if and only if there exists a function φ: $\Omega_0 \rightarrow \Omega'_0$ where $\Omega_0 \in \Im, \Omega'_0 \in \Im'$, $P(\Omega_0) = P(\Omega'_0) = 1$, and φ satisfies the following conditions:

(i) φ is $1-1$,

(ii) If $A \subset \Omega_0$ and $A' = \varphi A$ then $A \in \Im$ iff $A' \in \Im'$, and if $A \in \Im$

$$P(A) = P'(A'),$$

(iii) $T\Omega_0 \subseteq \Omega_0$ and $T'\Omega'_0 \subseteq \Omega'_0$,

(iv) For any ω in Ω_0

$$\varphi(T\omega) = T'\varphi(\omega).$$

I emphasize that the isomorphism in the measure-theoretic sense of two markets, two elections, or a market and an election seems at the right level of abstraction. The isomorphism expresses that the two structures have the same degree of uncertainty and thus the same structural freedom, even though they differ considerably in other characteristics. The fundamental point is that our conception of freedom needs to be at a rather high level of abstraction in order to be conceptually useful. It would be of little use if we ended up by making the freedom of each market or election *sui generis*, and thus not comparable to any other. What we should have is a methodology for comparing degrees of freedom. The isomorphism in a measure-theoretic sense of two stationary stochastic processes provides the important step of giving us a meaningful basis in terms of uncertainty for judging equivalence in freedom. Note why this is so. The φ function mapping one process into another is measure-preserving, so there is a structural isomorphism between corresponding events of the two processes such that they have the same probability. It is precisely the fact that the mapping carries events into events of the same probability that supports the claim that isomorphism in the measure-theoretic sense represents equivalence of uncertainty, and thus, of freedom of markets or elections.

On the other hand, it is equally important to note that isomorphism in the measure-theoretic sense of two stochastic markets only means isomorphism in the structure of uncertainty, as I have called it. Such isomorphism does not imply observational equivalence, nor would we want it to. For example, a Bernoulli market and a Markov market with strong dependence from one period to the next can be isomorphic in the measure-theoretic sense but easily distinguishable by a chi-square test for dependence. What we want to be able to say about these two markets is that they are equivalent in terms of freedom, but clearly different in other respects.

To show how recent fundamental results are about the relation between entropy rate and measure-theoretic isomorphism, I note that it was an open question in the 1950s whether the two finite-state discrete Bernoulli processes $B(\frac{1}{2}, \frac{1}{2})$ and $B(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ are isomorphic. (The notation here should be clear, as explained earlier; $B(\frac{1}{2}, \frac{1}{2})$ means that the probability for the Bernoulli process with two outcomes on each trial is that for each trial the probability of one alternative is $\frac{1}{2}$ and of the other $\frac{1}{2}$.) The following theorem clarified the situation.

Theorem 1 (Kolmogorov, 1958, 1959, and Sinai, 1959). *If two finite-state, discrete Bernoulli or Markov processes have different entropies, then they are not isomorphic in the measure-theoretic sense.*

Then the question became whether or not entropy is a complete invariant for measure-theoretic isomorphism. The following theorem was proved a few years later by Ornstein.

Theorem 2 (Ornstein, 1970). *If two finite-state, discrete Bernoulli processes have the same entropy rate then they are isomorphic in the measure-theoretic sense.*

This result was then soon easily extended.

Theorem 3 (Adler, Shields and Smorodinsky, 1972) *Any two irreducible, stationary, finite-state, discrete Markov processes are isomorphic in the measure-theoretic sense if and only if they have the same periodicity and the same entropy.*

We then obtain:

Corollary 1 *An irreducible, stationary, finite-state, discrete Markov process is isomorphic in the measure-theoretic sense to a finite-state, discrete Bernoulli process of the same entropy rate if and only if the Markov process is aperiodic.*

Given a stationary stochastic market or election the case is a good one for accepting entropy rate as an appropriate measure of freedom. To take advantage of the intuitions and results of ergodic theory this rather drastic abstraction has been used, a practice not uncommon in economics, but not to be commended. It is a task for the future to modify the theoretical framework to make it more empirically realistic, but still able to deal with markets or elections as dynamic processes over an extended period of time, not just in terms of a single cross section. (What is critical is approximate stationarity, and fortunately this can be statistically evaluated for the finite sequence of time periods available, a matter discussed in the next section.)

3 Learning, Uncertainty and Freedom: A Seeming Paradox

The paradox I have in mind runs along the following lines.

1. One of the primary functions of learning is to reduce uncertainty. Bayesian conditionalization and other forms of learning should in general reduce uncertainty, because of the increased knowledge that accompanies greater learning.

2. The reduction of uncertainty means a reduction in freedom.

3. Conclusion: learning reduces freedom.

This inference seems very unsatisfactory and is the reason for mentioning a *seeming* paradox.

But it is easy enough to give examples to show that the seeming paradox is not a real one. Implicit in the paradox is that as we get increased knowledge, we are always moving more toward a deterministic world. But this is not the case. Consider, for example, the person who becomes a very experienced player of certain kinds of gambling games. That person may decide that, by far, the most cost-free and efficient strategy is a mixed minimax strategy. The introduction of the randomness accompanying the mixed minimax strategy can, in fact, play the role of increasing uncertainty – uncertainty in outcome, not only for the person using the strategy, but in many cases, for all other players as well.

The second preconception that we must rid ourselves of is that as we gain knowledge, we always move toward a state of certainty, which in many cases means a state of certainty regarding true knowledge of the external world. But it is quite clear that this is not the case. As we learn more about meteorology, for instance, we abandon our naive beliefs that it is just a matter of blood, sweat and tears to correctly predict the weather for at least a week or two in advance. It is only with much sophisticated knowledge that we realize there is something intrinsically unstable and uncertain, from our standpoint, about the weather. It doesn't matter if we hold deterministic views of the world. Determinism does not imply predictability. How ever we view the world, deterministic or not, we are in a state of uncertainty regarding prediction of the weather. Moreover, we have, as we learn a great deal more about meteorology, ever greater confidence in our inability to make such predictions. We cannot eliminate the uncertainty we have about the future behavior of the weather once we go out even a few days in our predictions. Accurate predictions six months in advance are probably forever out of reach.

I now describe a simple learning model that will be the conceptual platform for various additional remarks about learning. Consider an experiment in which the subjects, that is, individuals in the experiment, make one of two responses and, on each trial, they are told which response was correct. Many variants on these restrictions can easily be studied, but the basic ideas can be illustrated in this simple framework, much used in Suppes and Atkinson (1960). A subject is given a sequence of trials. On each trial he makes either one of two responses, A_1 or A_2. Using boldface letters for random variables, we may thus define the *response random variable*:

$$\mathbf{A}_n(x) = \begin{cases} 1 & \text{if subject } x \text{ makes response } A_1 \text{ on trial } n, \\ 2 & \text{if subject } x \text{ makes response } A_2 \text{ on trial } n. \end{cases}$$

After x's response, the correct response is appropriately indicated to him. Indication of the correct response constitutes reinforcement. On each trial, exactly one of two reinforcing events, either E_1 or E_2, occurs. The occurrence of E_i means that A_i (for $i = 1, 2$) was the correct response. Thus we may define the *reinforcement random variable*:

$$\mathbf{E}_n(x) = \begin{cases} 1 & \text{if on trial } n \text{, reinforcement } E_1 \text{ occurred for subject } x, \\ 2 & \text{if on trial } n \text{, reinforcement } E_2 \text{ occurred for subject } x. \end{cases}$$

For those experiments in which the available stimuli are the same on all trials, we can use a model that dispenses with the concept of stimuli. In such a "pure" reinforcement model there is only one assumption: that the probability of a response on a given trial is a linear function of the probability of that response on the previous trial. A one-person experiment may be represented simply as a sequence $(\mathbf{A}_1, \mathbf{E}_1, \mathbf{A}_2, \mathbf{E}_2, \ldots, \mathbf{A}_n, \mathbf{E}_n, \ldots)$ of the response and reinforcement random variables defined above. Any sequence of values of these random variables represents a possible experimental outcome.

The linear theory is formulated for the probability of a response on trial $n+1$, *given* the entire preceding sequence of responses and reinforcements.[3] For this preceding sequence we use the notation x_n. Thus, x_n is a sequence of length $2n$ with 1's and 2's in the odd positions indicating responses A_1 and A_2, and 1's and 2's in the even positions indicating reinforcing events E_1 and E_2. The axioms of the linear theory are as follows:

Axiom L1. *If* $E_n = 1$ *and* $P(x_n) > 0$, *then*

$$P(A_{n+1} = 1 \mid x_n) = (1-\theta)\, P(A_n = 1 \mid x_{n-1}) + \theta.$$

Axiom L2. *If* $E_n = 2$ *and* $P(x_n) > 0$, *then*

$$P(A_{n+1} = 1 \mid x_n) = (1-\theta)\, P(A_n = 1 \mid x_{n-1}).$$

Here θ is the learning parameter.

Reinforcement is noncontingent when it does not depend on the subject's response. I consider here only the simple case of determinant noncontingent reinforcement. On each trial n

$$P(\mathbf{E}_n = 1 \mid \mathbf{A}_n, x_{n-1}) = \pi$$

independent of $\mathbf{A}_n$ and x_{n-1}. And

$$P(\mathbf{E}_n = 2 \mid \mathbf{A}_n, x_{n-1}) = 1 - \pi, \text{ where } 0 \leq \pi \leq 1.$$

For this noncontingent case we can derive from L1 and L2 the asymptotic mean result

$$\lim_{n\to\infty} P(A_n = 1) = \pi.$$

On the other hand, the expression for the variance of the Cesàro sum $\overline{A}_N$ for N trials at asymptote is more complicated. We shall not derive it here, but in Estes and Suppes (1959) it is shown to be

$$\operatorname{var}(\overline{A}_N) = \frac{\pi(1-\pi)}{(2-\theta)\,\theta}\left\{N\theta(4-3\theta) - 2(1-\theta)\left[1-(1-\theta)^N\right]\right\}$$

We shall also want to consider certain conditional probabilities at asymptote. Consider expressions of the form

$$\lim_{n\to\infty} P(A_{i,n+1} \mid E_{j,n} A_{k,n}) = P_\infty(A_{i,n+1} \mid E_{j,n} A_{k,n})$$

[3] In the language of stochastic processes, this means that we have not a Markov chain but a chain of infinite order.

Expressions for these quantities are given in Estes and Suppes (1959); we restate them here for the noncontingent case:

$$
\begin{aligned}
P_\infty\left(A_{1,n+1} \mid E_{1,n} A_{1,n}\right) &= (1-\theta)\, a+\theta, \\
P_\infty\left(A_{1,n+1} \mid E_{2,n} A_{1,n}\right) &= (1-\theta)\, a, \\
P_\infty\left(A_{2,n+1} \mid E_{1,n} A_{2,n}\right) &= (1-\theta)\, b, \\
P_\infty\left(A_{2,n+1} \mid E_{2,n} A_{2,n}\right) &= (1-\theta)\, b+\theta,
\end{aligned}
$$

where $a=[2\pi(1-\theta)+\theta]/(2-\theta)$ and $b=[2(1-\pi)(1-\theta)+\theta]/(2-\theta)$. Note that for $0<\pi, \theta<1$

$$P_\infty(A_1 \mid E_1 A_1) > P_\infty(A_1 \mid E_2 A_1)$$

and

$$P_\infty(A_2 \mid E_2 A_2) > P_\infty(A_2 \mid E_1 A_2).$$

We can easily derive other expressions. for example, the expression for $P_\infty\,(A_1 \mid E_1 A_1 E_1 A_1$ Namely,

$$
\begin{aligned}
& P\left(A_{1,n+1} \mid E_{1,n} A_{1,n} E_{1,n-1} A_{1,n-1}\right) P\left(E_{1,n} A_{1,n} E_{1,n-1} A_{1,n-1}\right) \\
&= \sum_{x_{n-2}} P\left(A_{1,n+1} E_{1,n} A_{1,n} E_{1,n-1} A_{1,n-1} x_{n-2}\right) \\
&= \pi^2 \sum_{x_{n-2}} P\left(A_{1,n+1} \mid E_{1,n} A_{1,n} E_{1,n-1} A_{1,n-1} x_{n-2}\right) \\
&\qquad P\left(A_{1,n} \mid E_{1,n-1} A_{1,n-1} x_{n-2}\right) \cdot P\left(A_{1,n-1} \mid x_{n-2}\right) P\left(x_{n-2}\right) \\
&= \pi^2 \sum_{x_{n-2}} [P\left(A_{1,n-1} \mid x_{n-2}\right)^3 (1-\theta)^3 + P\left(A_{1,n-1} \mid x_{n-2}\right)^2 \\
&\qquad \theta(1-\theta)(3-2\theta) + P\left(A_{1,n-1} \mid x_{n-2}\right) \theta^2 (2-\theta)] P\left(x_{n-2}\right).
\end{aligned}
$$

Hence. as $n \rightarrow \infty$,

$$
\begin{aligned}
& P_\infty\left(A_{1,n+1} \mid E_{1,n} A_{1,n} E_{1,n-1} A_{1,n-1}\right) \\
&= \frac{(1-\theta)^3 V_3+\theta(1-\theta)(3-2\theta) V_2+\theta^2(2-\theta)\pi}{(1-\theta) V_2+\theta\pi}.
\end{aligned}
$$

Where we have used raw moments of the following sort:

$$V_{i,n+1} = \sum_{x_n} P\left(A_{1,n+1} \mid x_n\right)^i P\left(x_n\right),$$

and

$$V_i = \lim_{n\rightarrow\infty} V_{i,n}$$

For the noncontingent case, it can easily be shown that

$$V_1 = \pi,$$

$$\begin{aligned} V_2 &= \pi^2 + \frac{\theta\pi(1-\pi)}{2-\theta}, \\ V_3 &= \pi^3 + \frac{\pi(1-\pi)\theta^3(1-2\pi)}{1-(1-\theta)^3} + \frac{3\theta(1-\pi)\pi^2}{2-\theta}. \end{aligned}$$

Now, it is not my purpose here to give a really thorough exposition of all these conditional probabilities, but there are a couple of important points I want to make.

First, the asymptotic behavior under this model of the individual is not at all Bayesian. If $\pi > \frac{1}{2}$ the model does not say you should always choose response 1, and if $\pi < \frac{1}{2}$ always choose response 2. Rather, the model leads to probability matching, as you can see, for the mean probability of response asymptotically. Thus, uncertainty remains in the subject's response at asymptote.

Even more interesting is the consideration of what happens with the conditional probabilities. The model is one that was much studied years ago with animals and it is set up for biological kinds of responses. The organism, operating in an environment it does not understand or know much about, is always on the watch for changes. Even one recent reinforcement of one side leads to an increase in the probability of response on that side. Two reinforcements of the same kind lead to a still greater reinforcement, etc. As I like to put it, the organism is ready at a moment's notice to engage in more learning and to adapt to a new situation. This is very much not the case with Bayesian models of learning, even though in principle it is something that can be built into Bayesian thought. Another way of putting it – such simple learning models, elementary though they are, are ready for change and ready for uncertainty in the environment. This asymptotic characteristic of biologically suitable learning models is fundamental and is a way of refuting the seeming paradox. Freedom does not disappear when the environment is uncertain, but is increased by better knowledge of that uncertainty. Remember in this connection the meteorological example.

A Markov chain is ergodic if and only if there exists a unique asymptotic distribution of the states independent of the initial distribution. This definition is easily generalized to chains of infinite order (Lamperti and Suppes, 1959). We can approximate the entropy of the chain of infinite order defined by θ and π asymptotically for the noncontingent case considered here.

So, corresponding to the earlier result for first-order Markov chains we have as a first-order approximation

$$H(\theta,\pi) = -\sum_{i,j} P_\infty(E_j A_{i'}) \sum P_\infty(A_i \mid E_j A_{i'}) \log P_\infty(A_i \mid E_j A_{i'}).$$

First, we note

$$\begin{aligned} P_\infty(E_1A_1) &= \pi^2 \\ P_\infty(E_1A_2) &= \pi(1-\pi) \\ P_\infty(E_2A_1) &= (1-\pi)\pi \\ P_\infty(E_2A_2) &= (1-\pi)^2 \end{aligned}$$

So, using the earlier results above for the first-order conditional probabilities at asymptote, we have:

$$\begin{aligned} H(\theta,\pi) \;=\; & -\{\pi^2[((1-\theta)a+\theta)\log((1-\theta)a+\theta) \\ & +(1-\theta)(1-a)\log(1-\theta)(1-a)] \\ & +\pi(1-\pi)[(1-(1-\theta)b)\,log\,(1-(1-\theta)b) \\ & +(1-\theta)b\log(1-\theta)b] \\ & +(1-\pi)\pi[(1-\theta)a\log(1-\theta)a \\ & +(1-(1-\theta)a)\log(1-(1-\theta)a)] \\ & +(1-\pi)^2[(1-\theta)(1-b)\log(1-\theta)(1-b) \\ & +((1-\theta)b+\theta)\log((1-\theta)b+\theta)]\} \end{aligned}$$

Although the equation for $H(\theta,\pi)$ is rather complicated, the extreme cases of $\theta=0$ and $\theta=1$ are simple and instructive. For $\theta=0$

$$H(0,\pi) = -[\pi\log\pi + (1-\pi)\log(1-\pi)],$$

which is just the entropy of the Bernoulli process $B(\pi,1-\pi)$ of the reinforcement scheme. Asymptotically in two senses, i.e., as $n\to\infty$ and $\theta\to 0$, the responses become themselves the Bernoulli process $B(\pi,1-\pi)$. This is an asymptotic result with respect to θ, for if $\theta=0$ there is no learning and the process is not ergodic, as is easily checked by looking at the learning axioms $L1$ and $L2$ for this special case. But with a very small positive θ we get a process whose uncertainty is quite close to that of the Bernoulli process of reinforcement.

In contrast. when $\theta=1, H(1,\pi)=0$, for the entropy rate goes to 0 as $\theta\to 1$. This is not surprising, for if $\theta=1$, we can predict with certainty each response. In the experiments reported in Suppes and Atkinson, (1960, Ch. 10), a good estimate of θ in a noncontingent reinforcement experiment with $\pi=0.6$ is $\theta=0.19$, so the entropy is fairly high. but, of course, lower than the Bernoulli process $B(.6,.4)$. It hardly needs commenting that $\theta=1$ is a bad biological strategy in an uncertain world. Conceptually it leads to asymptotic responses that predict the next future reinforcement will be just like the immediate past one. To illustrate how bad this could be, let reinforcements form a two-state stationary Markov chain with

$$P(\mathbf{E}_{n+1}=1 \mid \mathbf{E}_n=1)=\epsilon$$

$$P(\mathbf{E}_{n+1}=2 \mid \mathbf{E}_n=2)=\epsilon.$$

Then as $\epsilon\to 0$ no responses would be reinforced when $\theta=1$.

Finally, there is another point to be made about markets. What is the source of uncertainty in the market? From the standpoint of buyers, their feelings of uncertainty are generated in the competitive market by the efforts of sellers to compete with each other and offer the best product. This is a complicated many-dimensional problem and the effort at competition increases uncertainty in the consumer's or buyer's judgment as to which is the best product. The same goes for candidates in elections, so that in a competitive society, either politically or economically, uncertainty is the order of the day. The more we as buyers learn about products, the more uncertain we may often be as to which to choose. This is the kind of market truly competitive producers create. Such uncertainty guarantees that freedom remains in markets or in elections, and learning does not necessarily entail a sharp reduction of freedom.

References

Adler, R. L., Shields, P., and Smorodinsky, M. (1972). Irreducible Markov shifts. *The Annals of Mathematical Statistics*, **43**:3, 1027–1029.

Curry, B. and George, K. (1983). Industrial concentration: a survey. *The Journal of Industrial Economics*, **XXX**, 203-255.

Encaoua, D. and Jacquemin, A. (1980). Degree of Monopoly, Indices of Concentration and Threat of Entry. *International Economic Review*, **21**, 87–105.

Estes, W. K. and Suppes, P. (1959). Foundations of linear models. Chap.8 in *Studies in Mathematical Learning Theory*, R. R. Bush and W. K. Estes (Eds.), Stanford, California: Stanford University Press.

Foley, D. K. (1994). A statistical equilibrium theory of markets. *Journal of Economic Theory*. **62**, 321–345.

Hume, D. (1739/1888). *A Treatise of Human Nature*. Oxford: Selby-Bigge edition.

Kant, I. (1781/1787). *Critique of Pure Reason*. N. Kemp Smith, Trans. (1929/1965). New York: St. Martin's Press.

Kolmogorov, A. N. (1958). A new metric invariant of transient dynamical systems and automorphisms in Lebesgue space. *Dokl. Akad. Nauk. SSSR*, **119**, 861-864. (Russian) MR **21** ♯ 2035a.

Kolmogorov, A. N. (1959). Entropy per unit time as a metric invariant of automorphism. *Dokl. Akad. Nauk. SSSR*, **124**, 754-755. (Russian) MR **2** ♯ 2035b.

Lamperti, J. and Suppes, P. (1959) Chains of infinite order and their application to learning theory. *Pacific Journal of Mathematics*, **9**, 739–754.

Mill, J. S. (1859/1991) *On Liberty*. Oxford: Oxford University Press.

Ornstein, D. S. (1970). Bernoulli shifts with the same entropy are isomorphic. *Advances in Mathematics*, **4**, 337–352.

Sinai, Ya. G. (1959). On the notion of entropy of a dynamical system. *Dokl. Akad. Nauk SSSR* **124**, 768–771.

Suppes, P. (1996). The nature and measurement of freedom. *Social Choice and Welfare*, **13**:2, 183–200.

Suppes, P. and Atkinson, R. (1960). *Markov Learning Models for Multiperson Interactions*. Stanford, California: Stanford University Press.

Tirole, J. (1988). *The Theory of Industrial Organization*. Cambridge: MIT Press.

United States Bureau of the Census (1975). *Historical Statistics of the United States, Colonial Times to 1970, Bicentennial Edition, Part II*. Washington, D.C.