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Some Asymptotic Properties of Luce's Beta Learning Model*

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For application to various simple learning situations, R.D. Luce and his collaborators, R.R. Bush and E.H. Galanter, have considered a learning model in which the changes in probability of response from trial to trial are not linear functions of the probability of response on the preceding trial ([1], [6]). We restrict ourselves here to situations in which one of two responses, A_1 and A_2 , is made. Let p_n be the probability of response A_1 on trial n , and let E_1 be the event of reinforcing response A_1 , and E_2 the event of reinforcing response A_2 .

Luce's beta model is then characterized by the following transformations:**

if A_j and E_k occurred on trial n , then for $j = 1, 2$ and $k = 1, 2$,

$$(1) \quad p_{n+1} = \frac{p_n}{p_n + \beta_{jk}(1 - p_n)},$$

where $\beta_{jk} > 0$. (Generally, we want $\beta_{j1} < 1$ and $\beta_{j2} > 1$, in order to reflect the primary effects of reinforcement; moreover, we ordinarily assume that $\beta_{11} < \beta_{21} < \beta_{12} < \beta_{22}$.) We also assume throughout this paper that $0 \neq p_1 \neq 1$.

The most important fact about (1) is that the operators commute. For example, suppose in the first n trials there are b_1 occurrences of

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** A more general formulation is to be found in Luce [6].

A_1E_1 , b_2 occurrences of A_2E_1 , b_3 occurrences of A_1E_2 , b_4 occurrences of A_2E_2 , then it is easily shown that

$$(2) \quad p_{n+1} = \frac{p_1}{p_1 + \beta_{11} \frac{b_1}{p_1} + \beta_{21} \frac{b_2}{p_1} + \beta_{12} \frac{b_3}{1-p_1} + \beta_{22} \frac{b_4}{1-p_1}}$$

The point of the present paper is to study asymptotic properties of the beta model for certain standard probabilistic schedules of reinforcement. The methods of attack used by Karlin in [4] and by Lamperti and Suppes in [5] for linear learning models do not directly apply to the non-linear beta model.

The basis of our approach is to change the state space (the probability p_n is the state) from the unit interval to the whole real line in such a way that the transformations (1) become simply translations. The noncontingent case (the next section) then reduces to sums of independent random variables; the contingent cases can also be studied by "comparing" the resulting random walks with the case of sums of random variables. The probabilistic tool for this is developed in section 3 and applied in sections 4 and 5.

The general conclusion to be drawn from our results is that for all but one case of noncontingent reinforcement individual response probabilities are ultimately either zero or one, which is in marked contrast to corresponding results for linear learning models. Absorption at zero or one also occurs for many, but not all, cases of contingent reinforcement.

Noncontingent Reinforcement with Two Operators

If the probability of a reinforcement is independent of response and trial number, we have what is called simple noncontingent reinforcement. Let π be the probability of an E_1 reinforcement, and for simplicity let

$$(3) \quad \left\{ \begin{array}{l} \beta_{11} = \beta_{21} = \beta , \\ \beta_{12} = \beta_{22} = \gamma , \\ 0 < \beta < 1 , \\ \gamma > 1 . \end{array} \right.$$

We seek an expression for the asymptotic probability distribution of response probabilities in terms of the numbers π , β and γ .

Define the random variable η_n recursively as follows:

$$\eta_1 = \begin{cases} \beta & \text{with prob } \pi , \\ \gamma & \text{with prob } 1-\pi . \end{cases}$$
$$\eta_{n+1} = \begin{cases} \eta_n \beta & \text{with prob } \pi , \\ \eta_n \gamma & \text{with prob } 1-\pi . \end{cases}$$

Define now the random variable X_n as follows:

$$X_n = \log \eta_n .$$

Then obviously,

$$(4) \quad X_{n+1} = \begin{cases} X_n + \log \beta & \text{with prob } \pi, \\ X_n + \log \gamma & \text{with prob } 1-\pi. \end{cases}$$

It is clear from (4) and what has preceded that X_n is the sum of n independent identically distributed random variables Y_i defined by:

$$Y_i = \begin{cases} \log \beta & \text{with prob } \pi, \\ \log \gamma & \text{with prob } 1-\pi. \end{cases}$$

By the strong law of large numbers, we then have with probability one as $n \rightarrow \infty$

$$(5) \quad \begin{aligned} X_n &\rightarrow \infty & \text{if } \pi \log \beta + (1-\pi)\log \gamma > 0, \\ X_n &\rightarrow -\infty & \text{if } \pi \log \beta + (1-\pi)\log \gamma < 0. \end{aligned}$$

Define now for any real number x

$$(6) \quad F_x(p_1) = \frac{p_1}{p_1 + e^x(1-p_1)}.$$

Then $p_{n+1} = F_{X_n}(p_1)$ for the sequence of reinforcements η_n , where

$X_n = \log \eta_n$. We may utilize these results to prove the following theorem.

Theorem 1. Let $c = \pi \log \beta + (1-\pi)\log \gamma$. Then with probability one

$$p_\infty = \begin{cases} 0 & \text{if } c > 0, \\ 1 & \text{if } c < 0. \end{cases}$$

If $c = 0$ then p_n oscillates between 0 and 1, so that with probability one

$$\limsup p_n = 1$$

$$\liminf p_n = 0 .$$

Despite this oscillation, there is a limiting distribution for p_n ; it is concentrated at 0 and 1 with equal probabilities $\frac{1}{2}$.

Proof: The results for $c > 0$ and $c < 0$ follow immediately from (5), (6), and the remark following. In case $c = 0$, note that $E(Y_1) = 0$. It is known [2] that the sums X_n are then "recurrent"; that is, they repeatedly take on values arbitrarily close to any "possible" value. In particular, X_n takes on repeatedly arbitrarily large and arbitrarily small values (with probability one), which upon recalling (6) proves the second statement. The third statement is a consequence of the central limit theorem, which implies that for any A , $\Pr(X_n > A)$ and $\Pr(X_n < -A)$ both converge to one-half as n increases. Again the assertion of the theorem follows from this fact and (6).

Two Theorems on Random Walks

Above we have seen that the two operator, noncontingent beta model gives rise to a Markov process on the real line such that from x the "moving particle" goes to $x+a$ or $x-b$ with (constant) probabilities ϕ and $1-\phi$. The contingent case leads to a similar process, except that the

transition probabilities become functions of x . The four operator model gives rise to a process with four possible transitions, from x to $x+a_i$, say, $i = 1,2,3,4$. In this section we shall obtain some simple results on processes of these sorts, in preparation for the study of the more general cases of the beta model. In the interest of clarity, we shall treat in full only the two operator case; the more general case can be handled in a very similar way, but the details are cumbersome. Our approach was suggested by the work of Hodges and Rosenblatt[3].

Let $\{X_n\}$ be a real Markov process such that if $X_n = x$,

$$(9) \quad X_{n+1} = \begin{cases} x+a & \text{with prob } \varphi(x) , \\ x-b & \text{with prob } 1-\varphi(x) , \end{cases}$$

where $0 < a,b,\varphi(x), 1-\varphi(x)$. Let $\{Y_n\}$ be another process of the same type (and with the same a and b) but with constants θ and $1-\theta$ as the transition probabilities in place of $\varphi(x)$ and $1-\varphi(x)$. First we prove a

Lemma. If for all $x \geq M$ we have $\varphi(x) \geq \theta$, and if $\Pr(Y_n \rightarrow +\infty) > 0$, then $\Pr(X_n \rightarrow +\infty) > 0$. If, on the other hand, for $x \geq M$ $\varphi(x) \leq \theta$ and if $\Pr(Y_n \rightarrow +\infty) = 0$, then $\Pr(X_n \rightarrow +\infty) = 0$.

Proof. Let $\{\xi_n\}$ be a sequence of independent random variables, each uniformly distributed on $[0,1]$. The $\{X_n\}$ process will be referred to $\{\xi_n\}$ by letting

$$(10) \quad X_{n+1} = \begin{cases} X_n + a & \text{if } \xi_{n+1} \leq \varphi(X_n), \\ X_n - b & \text{otherwise.} \end{cases}$$

This does lead to the transition law (9) as may easily be seen. The $\{Y_n\}$ process can be linked to $\{X_n\}$ by referring it after the manner of (10) to the same sequence $\{\xi_n\}$, so that $Y_{n+1} = Y_n + a$ if and only if $\xi_{n+1} \leq \theta$.

Choose $Y_0 > M$. Whatever the value of X_0 , since $\varphi(x) > 0$ there is positive probability that $X_m \geq Y_0$ for some m ; we can therefore assume $X_0 \geq Y_0$. We now assert that for those sequences $\{Y_n\}$ with the property that $Y_n \geq M$ for all n , the inequality $X_n \geq Y_n$ is also valid for all n . This follows from our construction "linking" the processes, and the assumption that $\varphi(x) \geq \theta$ for $x \geq M$; the transition $X_{n+1} = X_n - b$ and $Y_{n+1} = Y_n + a$ is impossible, so $X_n - Y_n$ can only increase.

To complete the proof, note that since $\Pr(Y_n \rightarrow +\infty)$ is positive, so is $\Pr(Y_n \rightarrow +\infty, Y_n \geq M \text{ for all } n)$. But the event $Y_n \rightarrow +\infty, Y_n \geq M \text{ for all } n$ may be considered as a set S in the sample-space of the sequence $\{\xi_n\}$; S is a set of positive probability, and is contained in the set $X_n \rightarrow \infty$ since on S , $X_n \geq Y_n$ and $Y_n \rightarrow \infty$. Hence $\Pr(X_n \rightarrow +\infty) > 0$. The second part of the lemma is proved in a very similar way, using the same construction linking $\{X_n\}$ and $\{Y_n\}$.

Theorem 2. Let $b/(a+b) = c$, and suppose that

$$(11) \quad \lim_{x \rightarrow +\infty} \varphi(x) = \alpha \quad \text{and} \quad \lim_{x \rightarrow -\infty} \varphi(x) = \beta$$

exist. Then if $\alpha < c$ and $\beta > c$, we have

$$(12) \quad \Pr(\limsup X_n = +\infty, \liminf X_n = -\infty) = 1 \quad (\{X_n\} \text{ is "recurrent"}),$$

while if $\alpha < (>)c$ and $\beta < (>)c$, then

$$(13) \quad \Pr(X_n \rightarrow -\infty (+\infty)) = 1.$$

Finally, if $\alpha > c$ and $\beta < c$,

$$(14) \quad \Pr(X_n \rightarrow +\infty) = \delta, \quad \Pr(X_n \rightarrow -\infty) = 1 - \delta$$

for some $0 < \delta < 1$.

Proof. Suppose, for instance, that $\alpha < c$. Let $\{Y_n\}$ (as in the lemma) be a process with constant transition probabilities θ and $1 - \theta$ where $\alpha < \theta < c$. The $\{Y_n\}$ process may be regarded as sums of random variables:

$$(15) \quad Y_n = Y_0 + \sum_{i=1}^n Z_i, \quad \text{where } \Pr(Z_i = a) = \theta \text{ and } \Pr(Z_i = -b) = 1 - \theta.$$

But $E(Z_i) = a\theta - b(1 - \theta) < 0$, since $\theta < c$; this implies that

$\Pr(Y_n \rightarrow -\infty) = 1$ by the law of large numbers. From the lemma we then have $\Pr(X_n \rightarrow +\infty) = 0$.

Similarly, if $\alpha > c$ it follows that $\Pr(X_n \rightarrow +\infty) > 0$. Since the lemma also holds for convergence to $-\infty$ (with φ and θ replaced by $1 - \varphi$ and $1 - \theta$), we obtain in the same way that $\beta < c$ makes $\Pr(X_n \rightarrow -\infty) > 0$, while if $\beta > c$ this probability is zero.

Consider the case when $\alpha < c$ and $\beta < c$; there is then positive probability of absorption at $-\infty$, but not at $+\infty$. It is not hard to see that $X_n \rightarrow -\infty$ with probability one; the idea is roughly as follows: Since $X_n \not\rightarrow +\infty$, we have $X_n \leq N$ infinitely often with probability arbitrarily close to 1 for some N . Now the probability that from or to the left of N the random walk goes and remains to the left of $N-M$ must be positive since $\Pr(X_n \rightarrow -\infty) > 0$. But in an infinite sequence of not necessarily independent trials, an event whose probability on each trial is bounded away from zero is certain to occur. Hence for any M , the random walk will eventually become and remain to the left of $N-M$, and therefore $X_n \rightarrow -\infty$ with probability arbitrarily close to 1 (and so equal to one). The other cases are similar; one can think of $\alpha > c$ or $\alpha < c$ as the conditions under which $+\infty$ is an absorbing or reflecting barrier, etc., and the process behaves accordingly.

Finally we describe the generalization to the four operator case. Let $\{X_n\}$ be a real Markov process such that if $X_n = x$, then

$$(17) \quad X_{n+1} = x + a_i \quad \text{with probability } \phi_i(x),$$

where $a_1, a_2 > 0 > a_3, a_4$, and $\phi_i(x) > 0$. Suppose

$$(18) \quad \lim_{x \rightarrow +\infty} \phi_i(x) = \alpha_i \quad \text{and} \quad \lim_{x \rightarrow -\infty} \phi_i(x) = \beta_i$$

exist, and let $\mu_+ = \sum_{i=1}^4 a_i \alpha_i$ and $\mu_- = \sum_{i=1}^4 a_i \beta_i$.

By methods entirely similar to those used above, but rather more involved, it is possible to prove

Theorem 3. For the process $\{X_n\}$ described above, if $\mu_+ < 0$ and $\mu_- > 0$ then (12) holds; if $\mu_+ < (>)0$ and $\mu_- < (>)0$ then (13) applies; while if $\mu_+ > 0$ and $\mu_- < 0$, (14) is valid.

Contingent Reinforcement with Two Operators

If the probability of reinforcement depends only on the immediately preceding response (on the same trial), we have (simple) contingent reinforcement. Let $\Pr(E_1 | A_1) = \pi_1$ and $\Pr(E_1 | A_2) = \pi_2$, and let the two operators β and γ be specified as in (3). Using (6), we define the random variable X_n recursively (note that we put $\log \gamma$ first, since $\log \gamma > 0$ and $\log \beta < 0$, in order most directly to apply Theorem 2):

$$(19) \quad X_{n+1} = \begin{cases} X_n + \log \gamma & \text{with prob } F_{X_n}(p_1)(1-\pi_1) + (1-F_{X_n}(p_1))(1-\pi_2) = \varphi(X_n), \\ X_n + \log \beta & \text{with prob } 1 - \varphi(X_n). \end{cases}$$

We observe that

$$(20) \quad \lim_{x \rightarrow +\infty} \varphi(x) = 1 - \pi_2 \quad \text{and} \quad \lim_{x \rightarrow -\infty} \varphi(x) = 1 - \pi_1.$$

Combining (20) and Theorem 2, we then have immediately:

Theorem 4. For the contingent case of the two-operator model,
let $c = -\log \beta / \log \gamma / \beta$. Then with probability one

(i) if $1 - \pi_2 < c$ and $1 - \pi_1 > c$ then

$$\limsup_n p_n = 1 \quad \text{and} \quad \liminf_n p_n = 0 ,$$

(ii) if $1 - \pi_2 < c$ and $1 - \pi_1 < c$ then $p_\infty = 1$,

(iii) if $1 - \pi_2 > c$ and $1 - \pi_1 > c$ then $p_\infty = 0$.

Moreover,

(iv) if $1 - \pi_2 > c$ and $1 - \pi_1 < c$ then for some δ with $0 < \delta < 1$

$$\Pr(p_n \rightarrow 1) = \delta , \quad \Pr(p_n \rightarrow 0) = 1 - \delta .$$

The intuitive character of the distinction between the results expressed in (i) and (iv) of this theorem should be clear. If $1 - \pi_2 < c$ and $1 - \pi_1 > c$, then probability zero of an A_1 response and probability one of an A_1 response are both reflecting barriers, whereas if $1 - \pi_2 > c$ and $1 - \pi_1 < c$, they are both absorbing barriers.

It is also to be noticed that except when $1 - \pi_1 = c$ or $1 - \pi_2 = c$, Theorem 4 covers all values of β , γ , π_1 and π_2 for the contingent case. It can be shown that if $1 - \pi_1 = c$ (or $1 - \pi_2 = c$) then probability one (respectively zero) of an A_1 response is again a reflecting barrier.

Contingent Reinforcement with Four Operators

We want finally to apply Theorem 3 to the contingent case of the general four-operator model formulated in (1). Analogous to (19) we have:

$$(21) \quad X_{n+1} = \begin{cases} X_n + \log \beta_{22} & \text{with prob } (1-\pi_2)(1-F_{X_n}(p_1)) = \varphi_{22}(X_n) , \\ X_n + \log \beta_{12} & \text{with prob } (1-\pi_1)F_{X_n}(p_1) = \varphi_{12}(X_n) , \\ X_n + \log \beta_{21} & \text{with prob } \pi_2(1-F_{X_n}(p_1)) = \varphi_{21}(X_n) , \\ X_n + \log \beta_{11} & \text{with prob } \pi_1 F_{X_n}(p) = \varphi_{11}(X_n) . \end{cases}$$

And also,

$$(22) \quad \begin{cases} \lim_{x \rightarrow +\infty} \varphi_{22}(x) = 1-\pi_2 , & \lim_{x \rightarrow -\infty} \varphi_{22} = 0 , \\ \lim_{x \rightarrow +\infty} \varphi_{12}(x) = 0 , & \lim_{x \rightarrow -\infty} \varphi_{12}(x) = 1-\pi_1 , \\ \lim_{x \rightarrow +\infty} \varphi_{21}(x) = \pi_2 , & \lim_{x \rightarrow -\infty} \varphi_{21}(x) = 0 , \\ \lim_{x \rightarrow +\infty} \varphi_{11}(x) = 0 , & \lim_{x \rightarrow -\infty} \varphi_{11}(x) = \pi_1 . \end{cases}$$

Then

$$(23) \quad \mu_+ = \sum_{j,k} \log \beta_{jk} \lim_{x \rightarrow +\infty} \varphi_{jk}(x) = \pi_2 \log \beta_{21} + (1-\pi_2) \log \beta_{22} ,$$

and

$$(24) \quad \mu_- = \sum_{j,k} \log \beta_{jk} \lim_{x \rightarrow -\infty} \varphi_{jk}(x) = \pi_1 \log \beta_{11} + (1-\pi_1) \log \beta_{12} .$$

To apply Theorem 3 we need also to assume that $\beta_{22}, \beta_{12} > 1 > \beta_{21}, \beta_{11} > 0$.

On this assumption, and utilizing (23) and (24), we infer:

Theorem 5. For the contingent case of the four-operator model, we have the following results: with probability one

(i) if $\mu_+ < 0$ and $\mu_- > 0$ then

$$\limsup_n p_n = 1 \text{ and } \liminf_n p_n = 0 ,$$

(ii) if $\mu_+ < 0$ and $\mu_- < 0$ then $p_{\infty} = 1$,

(iii) if $\mu_+ > 0$ and $\mu_- > 0$ then $p_{\infty} = 0$;

and if $\mu_+ > 0$ and $\mu_- < 0$ then for some δ with $0 < \delta < 1$

$$\Pr(p_n \rightarrow 1) = \delta , \Pr(p_n \rightarrow 0) = 1 - \delta .$$

Specialization of this theorem to cover the noncontingent case is immediate.

References

- [1] Bush, R.R., E.H. Galanter, and Luce, R.D. Tests of the "beta model," Chapter 18, Mathematical studies in learning theory. Edited by R.R. Bush and W.K. Estes, Stanford University Press, 1959.
- [2] Chung, K.L. and Fuchs, W.H.J. On the distribution of values of sums of random variables. Mem. Amer. Math. Soc., 1951, 6, 1-12.
- [3] Hodges, J.L. and Rosenblatt, M. Recurrence time moments in random walks. Pac. J. of Math., 1953, 3, 127-136.
- [4] Karlin, S. Some random walks arising in learning models I. Pac. J. of Math., 1953, 3, 725-756.
- [5] Lamperti, J. and Suppes, P. Chains of infinite order and their application to learning theory. Pac. J. of Math. (to appear)
- [6] Luce, R.D. Individual choice behavior. New York: Wiley, 1959, forthcoming.

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