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NECESSARY AND SUFFICIENT CONDITIONS FOR EXISTENCE OF A UNIQUE MEASURE STRICTLY AGREEING WITH A QUALITATIVE PROBABILITY ORDERING*

1. CONCEPTUAL BACKGROUND

Let Ω be a nonempty set and let $\mathfrak{F}$ be an algebra of events on Ω, i.e., an algebra of sets on Ω. Let $\succcurlyeq$ be a qualitative ordering on $\mathfrak{F}$. The interpretation of $A \succcurlyeq B$ for two events A and B is that A is *at least as probable* as B. A (finitely additive) probability measure P on $\mathfrak{F}$ is *strictly agreeing* with the relation $\succcurlyeq$ if and only if, for any two events A and B in $\mathfrak{F}$,

$$P(A) \geqslant P(B) \text{ iff } A \succcurlyeq B.$$

A variety of conditions that guarantee the existence of a strictly agreeing measure is known. Without attempting a precise classification, the sets of conditions are of the following sorts: (i) sufficient but not necessary conditions for existence of a unique measure when the algebra of events is infinite (Koopman, 1940; Savage, 1954; Suppes, 1956); (ii) sufficient but not necessary conditions for uniqueness when the algebra of events is finite or infinite (Luce, 1967); sufficient but not necessary conditions for uniqueness when the algebra of events is finite (Suppes, 1969); (iv) necessary and sufficient conditions for existence of a not necessarily unique measure when the algebra of events is finite (Kraft, Pratt, & Seidenberg, 1959; Scott, 1964; Tversky, 1967). A rather detailed discussion of these various sets of conditions is to be found in Chapters 5 and 9 of Krantz, Luce, Suppes, and Tversky (1971).

The difficulties of giving reasonably simple conditions in terms of the qualitative ordering of events are exemplified by Luce's axiom, which is weaker than Koopman's equipartition axiom or Savage's related but somewhat stronger axiom. Luce's axiom is the following (Krantz *et al.*, 1971, p. 207):

> *For any events A, B, C, and D such that $A \cap B = \emptyset$, $A \succ C$,* and $B \succcurlyeq D$, *there exist events C', D', and E such that*

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(i) $E \approx A \cup B$;
(ii) $C' \cap D' = \emptyset$;
(iii) $C' \cup D' \subset E$;
(iv) $C' \approx C$ and $D' \approx D$.

Here $\succ$ is the strict ordering relation and $\approx$ the equivalence relation defined in terms of the weak ordering $\succcurlyeq$. The meaning of this axiom is complex and not easy to state in words. As we search for weaker axioms, closer to being necessary and not merely sufficient, the situation seems likely to get worse. The moral of the effort is that events are the wrong objects to consider. Some slightly richer concept is needed. Extension from one set of objects to a larger and richer set is a characteristic move in mathematics. The most familiar examples are extension of the rational numbers to the real numbers and extension of the real numbers to the complex numbers. As Kreisel has emphasized in several conversations, the introduction of auxiliary concepts is an indispensable practical move in solving significant problems in many domains of mathematics and science.

The main result of this article exemplifies how easily simplification can follow from the introduction of auxiliary concepts. In the present case the move is from an algebra of events to an algebra of extended indicator functions for the events. By this latter concept we mean the following. As before, let Ω be the set of possible outcomes and let $\mathfrak{F}$ be an algebra of events on Ω, i.e., $\mathfrak{F}$ is a nonempty family of subsets of Ω, and is closed under complementation and union, i.e., if A is in $\mathfrak{F}$, $\neg A$, the complement of A with respect to Ω, is in $\mathfrak{F}$, and if A and B are in $\mathfrak{F}$ then $A \cup B$ is in $\mathfrak{F}$. Let A^c be the indicator function (or characteristic function) of event A. This means that A^c is a function defined on Ω such that for any ω in Ω,

$$A^c(\omega) = \begin{cases} 1 \text{ if } \omega \in A \\ 0 \text{ if } \omega \notin A. \end{cases}$$

The algebra $\mathfrak{F}^*$ of *extended* indicator functions relative to $\mathfrak{F}$ is then just the smallest semigroup (under function addition) containing the indicator functions of all events in $\mathfrak{F}$. In other words, $\mathfrak{F}^*$ is the intersection of all sets with the property that if A is in $\mathfrak{F}$ then A^c is in $\mathfrak{F}^*$, and if A^* and B^* are in $\mathfrak{F}^*$, then $A^* + B^*$ is in $\mathfrak{F}^*$. It is easy to show that any function A^* in $\mathfrak{F}^*$ is an integer-valued function defined on Ω. It is the extension from indicator functions to integer-valued functions that justifies calling the elements of $\mathfrak{F}^*$ extended indicator functions.

The qualitative probability ordering must be extended from $\mathfrak{F}$ to $\mathfrak{F}^*$, and the intuitive justification of this extension must be considered. Let A^* and B^* be two extended indicator functions in $\mathfrak{F}^*$. Then, to have $A^* \succcurlyeq B^*$ is to have the expected value of A^* equal to or greater than the expected value of B^*. As should be clear, extended indicator functions are just random variables of a restricted sort. The qualitative comparison is now not one about the probable occurrences of events, but about the expected value of certain restricted random variables. The indicator functions themselves form, of course, a still more restricted class of random variables, but qualitative comparison of their expected values is conceptually identical to qualitative comparison of the probable occurrences of events.

There is more than one way to think about the qualitative comparison of the expected value of extended indicator functions, and so it is useful to consider several examples.

(i) Suppose Smith is considering two locations to fly to for a weekend vacation. Let A_i be the event of sunny weather at location i and B_i be the the event of warm weather at location i. The qualitative comparison Smith is interested in is the expected value of $A_1^c + B_1^c$ *versus* the expected value of $A_2^c + B_2^c$. It is natural to insist that the utility of the outcomes has been too simplified by the sums $A_i^c + B_i^c$. The proper response is that the expected values of the two functions are being compared as a matter of belief, not value or utility. Thus it would seem quite natural to bet that the expected value of $A_1^c + B_1^c$ will be greater than that of $A_2^c + B_2^c$, no matter how one feels about the relative desirability of sunny *versus* warm weather. Put another way, within the context of decision theory, extended indicator functions are being used to construct the subjective probability measure, not the measurement of utility. In this context it is worth recalling the importance of certain special decision functions – the gambles – in Savage's theory.

(ii) Consider a particular population of n individuals, numbered $1, \ldots, n$. Let A_i be the event of individual i going to Hawaii for a vacation this year, and let B_i be the event of individual i going to Acapulco. Then define

$$A^* = \sum_{i=1}^{n} A_i \quad \text{and} \quad B^* = \sum_{i=1}^{n} B_i.$$

Obviously A^* and B^* are extended indicator functions – we have left implicit the underlying set Ω. It is meaningful and quite natural to qualitatively compare the expected values of A^* and B^*. Presumably such comparisons are in fact of definite significance to travel agents, airlines, and the like.

We believe that such qualitative comparisons of expected value are natural in many other contexts as well. What the main theorem of this article shows is that very simple necessary and sufficient conditions on the qualitative comparison of extended indicator functions guarantee existence of a unique, strictly agreeing, finitely additive measure, whether the set Ω of possible outcomes is finite or infinite. The proof of the theorem, it should be mentioned, depends directly upon the theory of extensive measurement developed in Chapter 3 of Krantz *et al.* (1971).

2. FORMAL DEVELOPMENTS

The axioms are embodied in the definition of a qualitative algebra of extended indicator functions. Several points of notation need to be noted. First, Ω^c and $\emptyset^c$ are the indicator or characteristic functions of the set Ω of possible outcomes and the empty set $\emptyset$, respectively. Second, the notation nA^* for a function in $\mathfrak{F}^*$ is just the standard notation for the (functional) sum of A^* with itself n times. Third, the same notation is used for the ordering relation on $\mathfrak{F}$ and $\mathfrak{F}^*$, because the one on $\mathfrak{F}^*$ is an extension of the one on $\mathfrak{F}$: for A and B in $\mathfrak{F}$,

$$A \succcurlyeq B \text{ iff } A^c \succcurlyeq B^c.$$

Finally, the strict ordering relation $\succ$ is defined in the usual way: $A^* \succ B^*$ iff $A^* \succcurlyeq B^*$ and not $B^* \succcurlyeq A^*$.

DEFINITION. *Let Ω be a nonempty set, let $\mathfrak{F}$ be an algebra of sets on Ω, and let $\succcurlyeq$ be a binary relation on $\mathfrak{F}^*$, the algebra of extended indicator functions relative to $\mathfrak{F}$. Then the qualitative algebra $(\Omega, \mathfrak{F}^*, \succcurlyeq)$ is* qualitatively satisfactory *if and only if the following axioms are satisfied for every A^*, B^*, and C^* in $\mathfrak{F}^*$*:

Axiom 1. *The relation $\succcurlyeq$ is a weak ordering of $\mathfrak{F}^*$*;

Axiom 2. $\Omega^c \succ \emptyset^c$;

Axiom 3. $A^* \succcurlyeq \emptyset^c$;

Axiom 4. $A^* \succcurlyeq B^*$ *iff* $A^* + C^* \succcurlyeq B^* + C^*$;

Axiom 5. *If* $A^* \succ B^*$ *then for every* C^* *and* D^* *in* $\mathfrak{F}^*$ *there is a positive integer n such that*

$$nA^* + C^* \succcurlyeq nB^* + D^*.$$

These axioms should seem familiar from the literature on qualitative probability. Note that Axiom 4 is the additivity axiom that closely resembles de Finetti's additivity axiom for events: *If* $A \cap C = B \cap C = \emptyset$, *then* $A \succcurlyeq B$ *iff* $A \cup C \succcurlyeq B \cup C$. As we move from events to extended indicator functions, functional addition replaces union of sets. What is formally of importance about this move is seen already in the exact formulation of Axiom 4. The additivity of the extended indicator functions is unconditional – there is no restriction corresponding to $A \cap C = B \cap C = \emptyset$. The absence of this restriction has far-reaching formal consequences in permitting us to apply without any real modification the general theory of extensive measurement. Axiom 5 has, in fact, the exact form of the Archimedean axiom used in Krantz *et al.* (1971, p. 73) in giving necessary and sufficient conditions for extensive measurement. Discussion of why the formally simpler axiom – *if* $A^* \succcurlyeq B^* \succ \emptyset^c$ *then there is an n such that* $nB^* \succcurlyeq A^*$ – is not precisely satisfactory in giving necessary and sufficient axioms will be found there.

We are now in a position to formulate the theorem that is paraphrased in the title of the article.

THEOREM. *Let* Ω *be a nonempty set, let* $\mathfrak{F}$ *be an algebra of sets on* Ω, *and let* $\succcurlyeq$ *be a binary relation on* $\mathfrak{F}$. *Then a necessary and sufficient condition that there exist a* [redacted] *strictly agreeing probability measure on* $\mathfrak{F}$ *is that there is an extension of* $\succcurlyeq$ *from* $\mathfrak{F}$ *to* $\mathfrak{F}^*$ *such that the qualitative algebra of extended indicator functions* $(\Omega, \mathfrak{F}^*, \succcurlyeq)$ *is qualitatively satisfactory.*

Proof. As already indicated, the main tool used in the proof is from the theory of extensive measurement: necessary and sufficient conditions for existence of a numerical representation, as given in Krantz *et al.* (1971, pp. 73–74, Theorem 1). More precisely, let A be a nonempty set, $\succcurlyeq$ a binary relation on A, and $\circ$ a binary operation closed on A. Then there exists a numerical function φ on A unique up to a positive similarity transformation (i.e., multiplication by a positive real number) such that for a and b in A

(i) $a \succcurlyeq b$ iff $\varphi(a) \geqslant \varphi(b)$,

(ii) $\varphi(a \circ b) = \varphi(a) + \varphi(b)$

if and only if the following four axioms are satisfied for all a, b, c, and d in A:

E1. The relation $\succcurlyeq$ is a weak ordering of A;

E2. $a \circ (b \circ c) \approx (a \circ b) \circ c$, where $\approx$ is the equivalence relation defined in terms of $\succcurlyeq$;

E3. $a \succcurlyeq b$ iff $a \circ c \succcurlyeq b \circ c$ iff $c \circ a \succcurlyeq c \circ b$;

E4. If $a \succ b$ then for any c and d in A there is a positive integer n such that $na \circ c \succcurlyeq nb \circ d$, where na is defined inductively.

It is easy to check that qualitatively satisfactory algebras of extended indicator functions as defined above satisfy these four axioms for extensive measurement structures. First, we note that functional addition is closed on $\mathcal{F}^*$. Second, Axiom 1 is identical to E1. Extensive Axiom E2 follows immediately from the associative property of numerical functional addition: For any A^*, B^*, and C^* in $\mathcal{F}^*$

$$A^* + (B^* + C^*) = (A^* + B^*) + C^*,$$

and so we have not just equivalence but identity. Axiom E3 follows from Axiom 4 and the fact that numerical functional addition is commutative. Finally, E4 follows from the essentially identical Axiom 5.

Thus, for any qualitatively satisfactory algebra $(\Omega, \mathcal{F}^*, \succcurlyeq)$ we can infer there is a numerical function φ on Ω such that for A^* and B^* in $\mathcal{F}^*$

(i) $A^* \succcurlyeq B^*$ iff $\varphi(A^*) \geqslant \varphi(B^*)$

(ii) $\varphi(A^* + B^*) = \varphi(A^*) + \varphi(B^*)$,

and φ is unique up to a positive similarity transformation.

Second, since for every A^* in $\mathcal{F}^*$

$$A^* + \emptyset^c = A^*,$$

we have at once from (ii)

$$\varphi(\emptyset^c) = 0.$$

as we know, the theory of extended indicator functions has not previously played a noticeable role in the theory of qualitative probability.

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NOTE

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