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Stochastic Learning Theories For A Response Continuum With Non-Determinate Reinforcement.*

Patrick Suppes and Joseph Zinnes

Introduction

Recent extensions of the stochastic learning theories (Suppes [1], [2]) to a continuum of responses have dealt exclusively with determinate reinforcement schedules. Determinate reinforcement conditions refer to experimental tasks in which the subject is informed of the "correct" response on each trial or, in the case of animal subjects, to experiments using a correction procedure. In the present paper further extension to the non-determinate case is considered.

Attempts to extend the finite stochastic theories--finite, that is, with regard to the number of response alternatives considered--to non-determinate problems have tended to be unsatisfactory for several reasons: 1) The assumptions have tended to be somewhat unnatural (e.g., the assumption that the decrement to one response is distributed to the remaining responses in an amount proportional to the probabilities associated with these responses), 2) the mathematics is unwieldy due to awkward expressions or a large number of simultaneous equations, and 3) the conclusions derived are generally rather weak.

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One of the unexpected gains in treating the non-determinate and the response continuum problem simultaneously has been immediate progress on all three points described above. More natural or familiar assumptions immediately suggest themselves. One can assume, for example, that the effects of reinforcement or non-reinforcement generalize symmetrically about the response performed. Furthermore, the mathematics involved, employing the more powerful methods of the calculus, is relatively straightforward and the resulting deductions include, for example, simple closed form expressions for the asymptotic response distribution.

Although the theory to be developed here is intended to have some generality, it will be helpful to have in mind one of the experiments underway at present (hereafter called the Target Experiment). In this experiment subjects are instructed to locate or "hit" an unseen target which is said to be located at some point on the circumference of a circle. The exact position of the target on each trial is determined by sampling from a fixed distribution defined over the circumference. If the subject's response lies within a specified distance of the target he is informed that he has a "hit" on that trial; otherwise a "miss". Since the subject is free to choose, at least theoretically, any point on the circumference of the circle, the response alternatives may be said to lie on a continuum. The non-determinate aspect of the experiment refers to the fact that the subject is not informed of the exact location of the target after a miss (or a hit).

The plan of this paper is to develop separately the general equations for two types of theories--continuous analogs of the finite linear and stimulus sampling theories--and to illustrate more specifically the character of the models by applying them to the Target Experiment. In these examples we shall use some arbitrary parameter values, although in any actual application of the model, these parameters would have to be estimated from the data.

The Linear Theory

The extension of the continuous linear theory to non-determinate conditions is more easily described by considering briefly the determinate case. Denoting the value of the response random variable on trial n by x_n and the value of the reinforcement variable by y_n ($a \leq x_n \leq b$ and $a \leq y_n \leq b$) the sequence of experimental outcomes preceeding the $(n+1)$st trial is described by the $2n$ dimensional vector

$$(1) \qquad s_n = (y_n, x_n, y_{n-1}, x_{n-1}, \ldots, y_1, x_1).$$

The response distribution on trial $n + 1$ which is of experimental interest can then be defined as the marginal distribution obtained by integrating over the $2n$ dimensions. In particular, if $j_{n+1}(x, y_n, x_n, \ldots, y_1, x_1)$ denotes the joint density function of the first $n+1$ responses and n reinforcements, then the response density of the $(n+1)$st response, $r_{n+1}(x)$, is defined as

(2) $$r_{n+1}(x) = \int_a^b \cdots \int_a^b j_{n+1}(x, y_n, x_n, \ldots, y_1, x_1)\, dy_n\, dx_n \cdots dy_1 dx_1$$

which for simplicity is written as

(3) $$r_{n+1}(x) = \int_a^b j_{n+1}(x, s_n)\, ds_n .$$

In (2) it has been assumed that both reinforcement and response variables are continuous variables. Although under determinate reinforcement conditions one could assume a discrete reinforcement variable (and hence a discrete reinforcement distribution) it is more natural or perhaps merely more interesting to assume that both the response and reinforcement variables are continuous. These conditions permit the usual one-to-one correspondence between response alternatives and the set of reinforcing events. For non-determinate conditions, on the other hand, simplicity is obtained by confining the reinforcement variable to two values, 1 or 0, a value of "1" denoting a correct or rewarded response and "0" an incorrect or unrewarded response. (More complicated non-determinate reinforcement schedules can be obtained by giving the subject additional information on incorrect trials.)

Using Y_n (or on occasion $Y_{j,n}$ where $Y_{0,n} = 0$ and $Y_{1,n} = 1$) as the discrete reinforcement variable the sequence of experimental outcomes, s_n, becomes

$$s_n = (Y_n, x_n, Y_{n-1}, x_{n-1}, \ldots, Y_1, x_1)$$

and the response density, $r_n(x)$ as defined in (2), then involves summing over n dimensions and integrating over the remaining n dimensions. (We shall continue, however, to employ the terminology of (3) for this case as well.)

As in the finite linear theories, the basic assumptions of the continuous theories are stated recursively, that is, as rules or laws indicating how the response densities (instead of response probabilities) change after each trial. There are two possible outcomes on each trial and hence two recursions to consider. If the response on trial n is correct (i.e., $Y_n = 1$) then it is assumed that

$$j_{n+1}(x|Y_{1,n}, x_n, s_{n-1}) = (1 - \theta)\, j_n(x|s_{n-1}) + \theta k_h(x, x_n). \tag{4}$$

The last term in (4), $k_h(x, x_n)$, requires some comment. In general, it will be assumed that this function is unimodal and symmetric about x_n so that the effect of this term in (4) is to spread out or generalize the reinforcement effects to neighboring points on the response continuum. To assume that the reinforcement is concentrated on the continuum at the point of the reinforced response is psychologically untenable--the subject cannot discriminate this well--and furthermore it leads to mathematically untractable expressions. It is perhaps simplest to think of the right side of (4) as merely involving the weighting of two distributions, one distribution of some complicated character reflecting the subject's past history, and the other a distribution which is unimodal and symmetric about the response reinforced on the given trial. The symmetric aspect of the k_h-distribution could of course be questioned when end effects are present, but it is reasonably compelling for circular or periodic continua.

When the response on trial n is incorrect we have an analogous expression:

$$(5) \quad j_{n+1}(x|Y_{0,n}, x_n, s_{n-1}) = (1 - \theta)\, j_n(x|s_{n-1}) + \theta k_m(x, x_n).$$

The exact nature of the k_m-function is not as obvious. Two possibilities will be considered in the application section following:

$$(1) \quad k_m = \frac{1}{b - a} \qquad \text{and} \qquad (2) \quad k_m = k_h(x, x_n - \pi) .$$

Assumption (1), the assumption of a uniform distribution, has the effect of "flattening out" the distribution based on the previous trials while assumption (2) treats reinforcement and non-reinforcement effects in a complementary manner. The maximum point of the k_h-distribution corresponds to the minimum point of the k_m-distribution and similarly for the minimum point of the k_h distribution. More detailed properties are discussed in the following section.

By combining (4) and (5) with (3) a simple recursion involving the response density, $r_n(x)$, may be obtained. The details of the derivation follow the procedure given in Suppes [1], so that here we need merely sketch the argument. Equation (3) can be written as follows, where any obvious limits of integration are omitted:

$$(6) \qquad r_{n+1}(x) = \iint \sum_i j_{n+1}(x,\ Y_{i,n},\ x_n,\ s_{n-1})\ dx_n\ ds_{n-1}$$

or more explicitly as

$$(7) \qquad r_{n+1}(x) = \iint j_{n+1}(x,\ Y_{0,n},\ x_n,\ s_{n-1})\ dx_n\ ds_{n-1} + \iint j_{n+1}(x,\ Y_{1,n},\ x_n,\ s_{n-1})\ dx_n\ ds_{n-1}\ .$$

Consider the second term on the right side of (7); rewriting this term using conditional probabilities

$$(8) \qquad \iint j_{n+1}(x|Y_{1,n},\ x_n,\ s_{n-1})\ j(Y_{1,n}|x_n,\ s_{n-1})\ j(x_n|s_{n-1})\ j(s_{n-1})\ dx_n\ ds_{n-1}$$

and substituting (4) into (8) gives

$$(9) \qquad \iint [(1-\theta)\ j_n(x|s_{n-1}) + \theta k_h(x,\ x_n)]\ j(Y_{1,n}|x_n,\ s_{n-1})\ j(x_n|s_{n-1}) \times j(s_{n-1})\ dx_n\ ds_{n-1}$$

$$= \iint (1-\theta)\ j_n(x|s_{n-1})\ j(s_{n-1})\ j(Y_{1,n}|x_n,\ s_{n-1})\ j(x_n|s_{n-1})\ dx_n ds_{n-1} + \iint \theta k_h(x,\ x_n)\ j(Y_{1,n}|x_n,\ s_{n-1})\ j(x_n,\ s_{n-1})\ dx_n\ ds_{n-1}$$

In a similar manner (5) may be substituted into the first term on the right side of (7) to give

$$(10) \quad \iint (1 - \theta)\, j_n(x|s_{n-1})\, j(s_{n-1})\, j(Y_{0,n}|x_n, s_{n-1})\, j(x_n|s_{n-1})\, dx_n\, ds_{n-1}$$

$$+ \iint \theta k_m(x, x_n)\, j(Y_{0,n}|x_n, s_{n-1})\, j(x_n, s_{n-1})\, dx_n\, ds_{n-1} \,.$$

The expressions in (9) and (10) may be combined and simplified by noting first of all that since

$$j(Y_{0,n}|x_n, s_{n-1}) + j(Y_{1,n}|x_n, s_{n-1}) = 1$$

the first terms of (9) and (10) can be combined and the integration over x_n carried out. Secondly, since Y_n does not depend on s_{n-1}, that is for example,

$$j(Y_{0,n}|x_n, s_{n-1}) = j(Y_{0,n}|x_n) \,,$$

the integration over s_{n-1} in each of the second terms in (9) and (10) can be performed. Thus combining these terms

$$(11) \quad r_{n+1}(x) = (1 - \theta)\int j_n(x|s_{n-1})\, j(s_{n-1})\, ds_{n-1}$$

$$+ \theta \int k_h(x, x_n)\, j(Y_{1,n}|x_n)\, j(x_n)\, dx_n$$

$$+ \theta \int k_m(x, x_n)\, j(Y_{0,n}|x_n)\, j(x_n)\, dx_n \,.$$

Equation (11) can be simplified further by replacing the first integral with $r_n(x)$ and to conform with the usual notation in the finite theories we shall replace $j(Y_{1,n}|x_n)$ with $\pi_{1,n}(x_n)$ and $j(Y_{0,n}|x_n)$ with $\pi_{0,n}(x_n)$; $j(x_n)$ is, of course, nothing but $r_n(x)$. Thus we have for the response density the following simple recursion:

(12) $$r_{n+1}(x) = (1 - \theta)\, r_n(x) + \theta \int k_h(x, x_n)\, \pi_{1,n}(x_n)\, r_n(x_n)\, dx_n + \theta \int k_m(x, x_n)\, \pi_{0,n}(x_n)\, r_n(x_n)\, dx_n .$$

From (12) the equation for the asymptotic response density, $r(x)$, follows directly:

(13) $$r(x) = \int k_h(x, x_n)\, \pi_1(x_n)\, r(x_n)\, dx_n + \int k_m(x, x_n)\, \pi_0(x_n)\, r(x_n)\, dx_n .$$

Equation (13) is a linear homogeneous integral equation in $r(x)$. The ease of solving (13) explicitly depends on the form of the function k_h, k_m, and π_1.

The functions π_1 and π_0 in (13) are determined by the experimenter so they can be considered as known functions here. In the Target Experiment these functions are determined indirectly. If the exact position of the target is denoted by y and its density by $f(y)$ then the probability of a "hit" on trial n given response x_n (i.e., $\pi_1(x_n)$) is given in terms of $f(y)$ by

$$\pi_1(x_n) = \int_{x_n-\alpha}^{x_n+\alpha} f(y)\, dy \ ,$$

where α, another experimenter-determined parameter, specifies the permissible range about the target or 2α the effective size of the target.

Special Case: Asymptotic Distributions In the Target Experiment

In this section we shall take certain one parameter functions for k_h, k_m and π_1 (or $f(y)$) and investigate their implications for the Target Experiment.

For the functions $k_h(x, x_n)$ and $f(y)$ we assume

(15) $$k_h(x, x_n) = C_j \cos^{2j}\left(\frac{x - x_n}{2}\right) \ , \qquad (j = 1, 2, \ldots)$$

(16) $$f(y) = C_i \cos^{2i}\left(\frac{y}{2}\right) \ , \qquad (i = 1, 2, \ldots)$$

where C_i (and C_j), the normalization factor, is given by

(17) $$C_i = \frac{2^{(2i-1)}\,(i!)^2}{\pi(2i)!} \ .$$

Some comment on the use of the cosine function raised to even powers as a density function is perhaps called for here. This function has three specific properties which make it particularly convenient to use in this context: the function is periodic; the indefinite integral has a closed form expression, and, most importantly, it leads to a degenerate kernel in (13).

(The kernel refers to the terms in each integral of (13) excluding the function $r(x)$. It is said to be degenerate if the two variables, x and x_n, can be separated; that is, for example, if functions g_i and h_i exist such that for some N

$$(18) \qquad \pi_1(x_n)\, k_h(x,\, x_n) = \sum_{i=1}^{N} g_i(x)\, h_i(x_n) \;.$$

The cosine function in (16) has this property since by the usual trigonometries identities

$$(19) \qquad \cos^{2i}\left(\frac{x - x_n}{2}\right) = \sum_{j=0}^{i} a_j \cos jx \cos jx_n + \sum_{j=1}^{i} b_j \sin jx \sin jx_n$$

where a_j and b_j are constants independent of x and x_n.) These three properties produce considerable simplification in solving (13) for $r(x)$.

Despite their somewhat unconventional appearance both density functions in (16) and (17) lead to reasonably conventional--we might almost say normal--distributions in the interval $(-\pi,\, \pi)$ and $(x - \pi,\, x + \pi)$, respectively. The distributions are unimodal in this interval, symmetric, and have two inflection points for $j > 1$ placed symmetrically about the mean. The value of the exponent of the cosine in each case determines the variance of the distribution. When, for example $j = 2$, the variance equals .79, and to compare this particular distribution with the normal distribution, .67 of the area lies within one sigma of the mean.

For the function $k_m(x, x_n)$ there are two possibilities (described previously) which we shall consider. They lead to what we term the Uniform Theory (or U-theory) and the Symmetric Theory (or S-theory).

Uniform Theory

This theory is characterized by the assumption

(20) $$k_m(x, x_n) = \frac{1}{2\pi} \ .$$

For illustrative purposes in this section it will be assumed that $j = 1$, i.e.

(21) $$k_h = \frac{1}{\pi} \cos^2\left(\frac{x - x_n}{2}\right),$$

although previous experimental work (Suppes and Frankman [3]) indicates that a more realistic assumption would involve a distribution with a variance approximately equal to .58 and this corresponds approximately to $j = 6$. For the reinforcement function $f(y)$ we shall take

(22) $$f(y) = \frac{1}{\pi} \cos^2(\frac{y}{2})$$

and therefore

(23) $$\pi_1(x_n) = \frac{1}{\pi} \int_{x_n-\alpha}^{x_n+\alpha} \cos^2 \frac{y}{2} \ dy = \frac{1}{\pi} (\alpha + \sin \alpha \cos x_n) \ .$$

Now to solve (13) for $r(x)$. Since the method of solving integral equations with degenerate kernels may not be familiar to the reader the details of the solution will be given. Some initial simplification of (13) is obtained by substituting (15) into (13), replacing $\pi_0(x_n)$ with $1 - \pi_1(x_n)$, and letting

(24) $$G = \frac{1}{2\pi} \int_{-\pi}^{\pi} \pi_1(x_n)\, r(x_n)\, dx_n$$

where G is a constant independent of x. Thus (13) reduces to

(25) $$r(x) = \frac{1}{2\pi} - G + \int_{-\pi}^{\pi} k_h(x, x_n)\, \pi_1(x_n)\, r(x_n)\, dx_n .$$

Consider the integral term in (25). First expanding (21)

(26) $$k_h(x, x_n) = \frac{1}{\pi} \cos^2 \left(\frac{x - x_n}{2}\right)$$
$$= \frac{1}{2\pi} (1 + \cos x \cos x_n + \sin x \sin x_n)$$

and then using (26) and (23) we have for this integral

(27) $$\int k_h(x, x_n)\, \pi_1(x_n)\, r(x_n)\, dx_n$$
$$= \int \frac{1}{2\pi} (1 + \cos x \cos x_n + \sin x \sin x_n) \frac{1}{\pi} (\alpha + \sin \alpha \cos x_n)\, r(x_n)\, dx_n$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \pi_1(x_n)\, r(x_n)\, dx_n$$
$$+ \frac{1}{2\pi^2} \int_{-\pi}^{\pi} [(\alpha \cos x) \cos x_n + (\sin \alpha \cos x) \cos^2 x_n$$
$$+ (\alpha \sin x) \sin x_n + (\sin \alpha \sin x) \sin x_n \cos x_n]\, r(x_n)\, dx_n$$

which simplifies to

$$\int k_h(x, x_n) \, \pi_1(x_n) \, r(x_n) \, dx_n = G + A \cos x + B \sin x \tag{28}$$

where the coefficients A and B, representing the definite integrals over x_n, are independent of x. Substituting (28) into (25) indicates that the density $r(x)$ must have the following form:

$$\begin{aligned} r(x) &= \frac{1}{2\pi} + G - (G + A \cos x + B \sin x) \\ &= \frac{1}{2\pi} + A \cos x + B \sin x. \end{aligned} \tag{29}$$

The coefficients in (29), A and B, are evaluated by obtaining a set of linear simultaneous equations as follows. The expression for $r(x)$ in (29) is placed into (27), replacing, first of all, x by x_n, and the resulting integrals evaluated. This gives a second expression for $r(x)$, viz.

$$\begin{aligned} r(x) &= \frac{1}{2\pi} + \frac{1}{2\pi^2} \left[\frac{1}{2} \sin \alpha \cos x + A\pi\alpha \cos x + B\pi\alpha \sin x\right] \\ &= \frac{1}{2\pi} + \left(\frac{\sin \alpha}{4\pi^2} + \frac{A\alpha}{2\pi}\right) \cos x + \left(\frac{B\alpha}{2\pi}\right) \sin x . \end{aligned} \tag{30}$$

Since both (29) and (30) must hold for all values of x we may equate the coefficients of $\cos x$ and $\sin x$ and thus obtain the equations

$$\left\{ \begin{aligned} A &= \left(\frac{\sin \alpha}{4\pi^2} + \frac{A\alpha}{2\pi}\right) \\ B &= \frac{B\alpha}{2\pi} \end{aligned} \right. \tag{31}$$

which gives

(32)
$$\begin{cases} A = \dfrac{\sin \alpha}{2\pi(2\pi - \alpha)} \\ B = 0 , \end{cases}$$

so that finally

(33)
$$r(x) = \frac{1}{2\pi} \left(1 + \frac{\sin \alpha}{2\pi - \alpha} \cos x\right) .$$

The asymptotic response distribution indicated by (33) is, as is expected, unimodal and symmetric about $x = 0$. Thus the means of the reinforcement distribution and response distribution coincide although the two distributions differ in variance. The variance of the response distribution, obtained directly from (33), is equal to

(34)
$$\sigma_u^2 = \frac{\pi^2}{3} - 2\left(\frac{\sin \alpha}{2\pi - \alpha}\right)$$

and the variance of the reinforcement distribution $f(y)$ equals

(35)
$$\sigma_f^2 = \frac{\pi^2}{3} - 2 = 1.29 .$$

Since α lies between 0 and π, it is clear from (34) and (35) that the response variance invariably exceeds the reinforcement variance. (In Table 1, σ_u^2 is given for various values of α. It is also instructive to compare these σ^2 values with the plot of $\pi_1(x_n)$ vs. x_n for corresponding values of α given in Figure 1.)

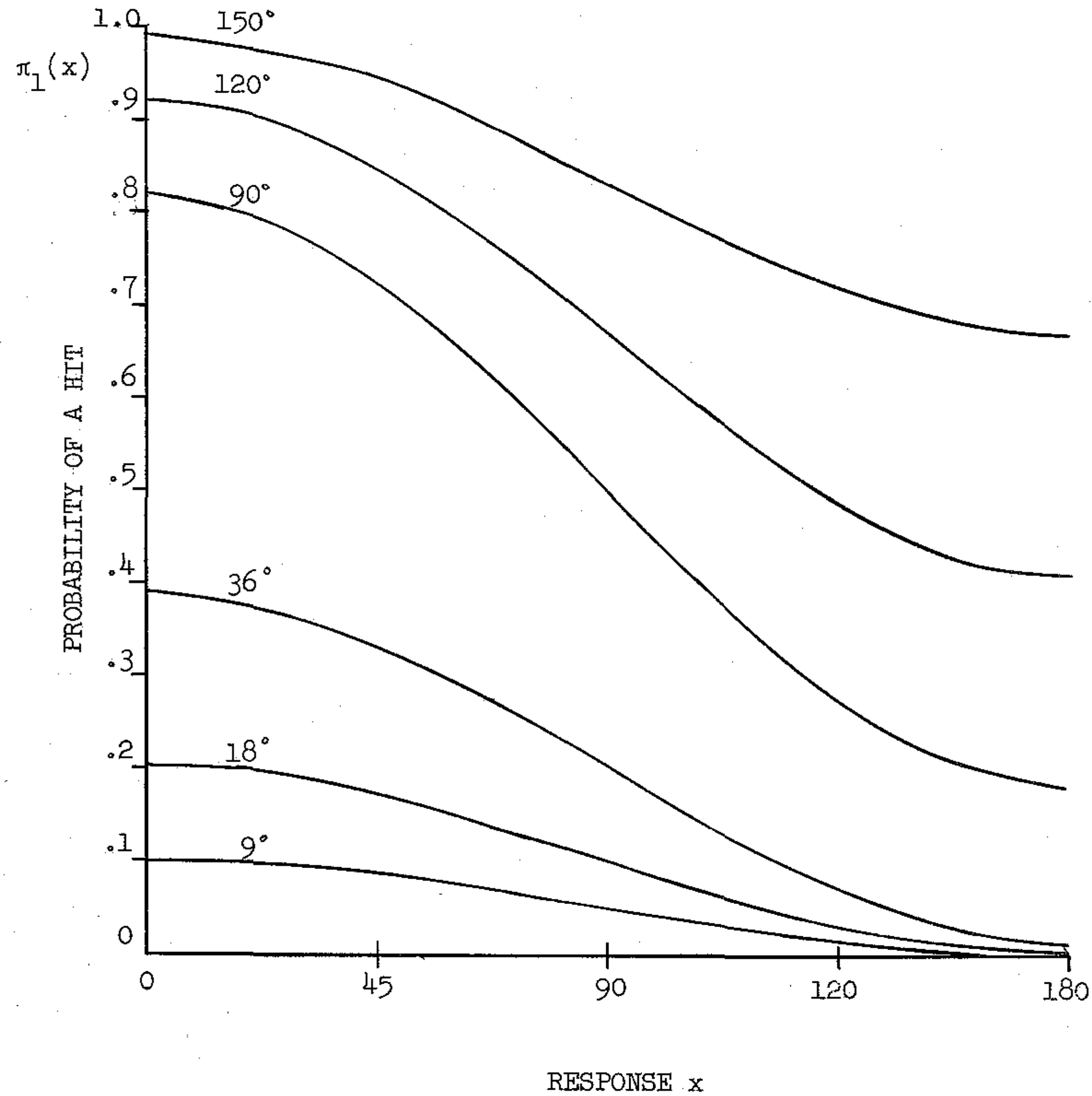

Figure 1. The probability of reinforcement for various values of the angle α. Due to their symmetry only half the curves are given. The reinforcement function $f(y)$ which has been assumed is given in (22).

TABLE 1

THEORETICAL VARIANCE OF THE ASYMPTOTIC RESPONSE DISTRIBUTION
FOR VARIOUS VALUES OF α

α	U-theory	S-theory	I-theory
0	3.290	3.290	2.290
9°	3.239	3.221	2.294
18°	3.186	3.149	2.306
36°	3.082	3.002	2.354
90°	2.866	2.653	2.653
120°	2.876	2.628	2.876
150°	3.017	2.812	3.099
180°	3.290	3.290	3.290

Symmetric Theory

In the S-theory, the previous assumption for the k_m-distribution is replaced by

$$k_m(x, x_n) = k_h(x, x_n - \pi) \tag{36}$$

while the remaining assumptions of the U-theory are applicable here as well. Under these assumptions (13) becomes

$$r(x) = \int k_h(x, x_n)\, \pi_1(x_n)\, r(x_n)\, dx_n + \int k_h(x, x_n - \pi)\, \pi_0(x_n)\, r(x_n)\, dx_n \tag{37}$$

To compare the predictive character of the uniform and symmetric models we shall solve (37) for the asymptotic response distribution, $r(x)$, taking for k_h and $f(y)$ the functions assumed previously in (21) and (22). The method of solution follows the same pattern outlined in solving (25) so that we shall omit the details here. From (37) then we have

$$r(x) = \frac{1}{2\pi}\left(1 + \frac{2 \sin \alpha}{3\pi - 2\alpha} \cos x\right), \tag{38}$$

which implies a response variance of

$$\sigma^2_{sym} = \frac{\pi^2}{3} - \frac{4 \sin \alpha}{3\pi - 2\alpha}. \tag{39}$$

The similarity of the density functions in (33) and (38) indicate with regard to the asymptotic response distribution that the uniform and symmetric models do not behave very differently. The two response distributions differ mainly in their variances, the variance associated with the uniform model being consistently larger. It will be noted in Table 1 that the response variance as a function of α has a minimum value in each theory, although the minimum variance occurs at different values of α. In the U-theory the response distribution $r(x)$ has a minimum variance when $\alpha = 1.79$ radians (103°) while in the S-theory the minimum variance occurs when $\alpha = 1.91$ radians (110°).

Although both theories lead to similar asymptotic response distributions they do have some different characteristics which are brought out by considering their sequential predictions. Unfortunately, however, the sequential statistics are not easily obtained with these linear models since simple recursions do not arise. To point up some notable differences between the uniform and symmetric models we shall consider in the next section the special case $\theta = 1$ for which simple results are possible.

Sequential Properties: $r_n(x|Y_{0,n-1})$

The response distribution on trial n, given a miss on trial $n - 1$, the sequential statistic to be considered here, is defined, in terms of the density functions, by

$$(40) \qquad r_{n+1}(x|Y_{0,n}) = \frac{j_{n+1}(x, Y_{0,n})}{P(Y_{0,n})}$$

where $P(Y_{0,n})$ represents the probability of a miss on trial $n - 1$. The numerator term in (40) is equal to the first integral in (7), that is,

$$(41) \qquad j_{n+1}(x, Y_{0,n}) = \iint j_{n+1}(x, Y_{0,n}, x_n, s_{n-1})\, dx_n\, ds_{n-1}$$

which, for general θ, gives

$$(42) \qquad \iint (1 - \theta)\, j_n(x, s_{n-1})\, \pi_0(x_n)\, j(x_n | s_{n-1})\, dx_n\, ds_{n-1} + \theta \int k_m(x, x_n)\, \pi_0(x_n)\, r_n(x_n)\, dx_n .$$

Since the first integral in (42) cannot be simplified further, we take $\theta = 1$, and thus have for the numerator term,

$$(43) \qquad j_{n+1}(x, Y_{0,n}) = \int k_m(x, x_n)\, \pi_0(x_n)\, r_n(x_n)\, dx_n .$$

The denominator term in (40) equals

$$(44) \qquad P(Y_{0,n}) = \int_{-\pi}^{\pi} r_n(x)\, \pi_0(x)\, dx$$

so that combining (43) and (44), we obtain the general expression sought:

$$(45) \qquad r_{n+1}(x | Y_{0,n}) = \frac{\int k_m(x, x_n)\, \pi_0(x_n)\, r_n(x_n)\, dx_n}{\int \pi_0(x)\, r_n(x)\, dx} .$$

Equation (45) is readily evaluated for the uniform model since $k_m(x, x_n)$ is a constant. Thus

$$(46) \qquad r_{n+1}(x|Y_{0,n}) = \frac{\frac{1}{2\pi}\int \pi_0(x_n)\ r_n(x_n)\ dx_n}{\int \pi_0(x)\ r(x)\ dx} = \frac{1}{2\pi}$$

which indicates that the response distribution following a miss in the uniform model is a uniform distribution independent of the trial number n.

For the symmetric model in which we assume equation (36), $r_{n+1}(x|Y_0)$ given in (45) depends on the response distribution of trial n so that in general it will change over trials. If, for example, the asymptotic response density given in (38) is inserted into (45), the conditional function which results is

$$(47) \qquad r(x|Y_{0,n}) = \frac{1}{2\pi}\left(1 + \frac{\pi \sin \alpha}{2(\pi - \alpha)\ (3\pi - 2\alpha) - 2 \sin^2 \alpha} \cos x\right) .$$

The distribution implied by (47) is unimodal in the interval $(-\pi, \pi)$ and symmetrical about $x = 0$ for all values of $\alpha < \pi$. Since (46) and (47) differ considerably the sequential statistic we are considering here forms a basis for distinguishing the two models empirically. It should be emphasized however that both (46) and (47) depend on the assumption $\theta = 1$ and this may be untenable or difficult to establish empirically for both models. The models to be considered in the next section do not have this defect.

Stimulus Sampling Theory

In the finite version of the stimulus sampling theory it is usually assumed that each element comprising the stimulus set is conditioned to at most one response at a given time. (An element may be in an unconditioned state, in which case the element is conditioned to no response.) The continuous version of the theory is obtained by relaxing this restriction and permitting an element to be conditioned to a range of responses. The conditioning state of an element is then represented by a density function $k(x,z)$ where z denotes the mean of the distribution and x a value of the response random variable. We shall further assume here, since we are mainly concerned with periodic continua, that the distribution implied by this density function is symmetric about the mean z. (In [2], $k(x,z)$ is called a smearing density.)

The sequence of events which takes place on a given trial is then conceptualized as follows. See [2] for a more detailed statement. A stimulus element is drawn by the subject from the set available on the given trial which, we shall assume in this paper, consists of exactly one element. In this special case, the state of the organism on each trial is given by a value of a single parameter, viz. the mean z; knowing the value of z (and, of course, the smearing density $k(x,z)$) we may determine the probability of the response x falling within any specified interval on the response continuum. In practice, however, the mean z is not known to the experimenter, and it is more meaningful to discuss the density of z on trial n (denoted by $g_n(z)$) and to define the response density,

$r_n(x)$, in terms of $g_n(z)$, that is,

$$r_n(x) = \int k(x,z)\, g_n(z)\, dz \quad (48)$$

It will be seen that it is more convenient to express the conditioning assumptions of the stimulus sampling theory in terms of recursions involving $g_n(z)$ rather than $r_n(x)$. Since equation (48) can be used to obtain $r_n(x)$ when $g_n(z)$ is known, this procedure will not result in any loss of generality.

Following the occurrence of the response x_n in the simple non-determinate case, four outcomes are possible. These outcomes are described by two dichotomous random variables, the reinforcement variable Y_n and a variable denoted by F_n (or $F_{i,n}$) which specifies whether the reinforcing event is "effective". When $Y_n = 1$ and $F_n = 1$ (or given $F_{1,n}$), i.e., when the subject has been informed he is "correct" and this reinforcing event is "effective", it is assumed that the mean of the k-distribution shifts to the point of the response, that is, $z_{n+1} = x_n$. To obtain the recursion which indicates how the density of z changes after each trial, we shall need the joint density of the events z_{n+1} (or x_n), $F_{1,n}$ and $Y_{1,n}$. If θ denotes the probability of $F_n = 1$, then

$$j_{n+1}(z, F_{1,n}, Y_{1,n}) = \theta\pi_1(z)\, r_n(z). \quad (49)$$

With probability $1 - \theta$ the reinforcement is not effective $(F_n = 0)$. Then the mean of the k-distribution does not shift, but remains fixed for the subsequent trial, that is, $z_{n+1} = z_n$. The joint density associated with these events is

$$j_{n+1}(z, F_{0,n}, Y_{1,n}) = (1 - \theta)\, g_n(z) \int k(x,z)\, \pi_1(x)\, dx. \tag{50}$$

It is further assumed that when the subject is incorrect $(Y_n = 0)$ and the "reinforcement" not effective $(F_n = 0)$ that $z_{n+1} = z_n$; thus

$$j_{n+1}(z, F_{0,n}, Y_{0,n}) = (1 - \theta)\, g_n(z) \int k(x,z)\, \pi_0(x)\, dx . \tag{51}$$

For the effect of the fourth possible outcome $(F_{1,n}$ and $Y_{0,n})$ on z_n, we shall consider three possible alternatives:

(I) $\quad z_{n+1} = z_n$

(S) $\quad z_{n+1} = x_n + \gamma$

(U) $\quad g(z_{n+1} | F_{1,n}, Y_{0,n}) = \frac{1}{2\pi}$

Assumptions (S) and (U), respectively lead, as will be seen, to stimulus sampling analogs of the symmetric model (when $\gamma = \pi$) and the uniform model discussed previously. Assumption (I) leads to a distinct theory we shall term the <u>Identity</u> (or I) <u>theory</u>. The joint densities corresponding to these three assumptions are as follows:

$$\text{(I)} \qquad j_{n+1}(z, F_{1,n}, Y_{0,n}) = \theta\, g_n(z) \int \pi_0(x)\, k(x,z)\, dx$$

$$\text{(52) (S)} \qquad j_{n+1}(z, F_{1,n}, Y_{0,n}) = \theta\, r_n(z - \gamma)\, \pi_0(z - \gamma)$$

$$\text{(U)} \qquad j_{n+1}(z, F_{1,n}, Y_{0,n}) = \frac{\theta}{2\pi} \int \pi_0(x)\, r_n(x)\, dx \;.$$

By combining (49), (50), (51) and one of the equations given in (52) a recursion for $g_n(z)$ may be obtained. We shall consider separately each of the three possibilities.

Identity Theory.

Equations (50) and (51) may be combined and simplified immediately to give

$$\text{(53)} \qquad \begin{aligned} j_{n+1}(z, F_{0,n}) &= (1 - \theta)\, g_n(z) \int k(x,z)\, (\pi_0(x) + \pi_1(x))\, dx \\ &= (1 - \theta)\, g_n(z) \end{aligned}$$

Combining (53) with (49) and (52I) gives the recursion

$$\text{(54)} \qquad g_{n+1}(z) = (1 - \theta)\, g_n(z) + \theta[\pi_1(z)\, r_n(z) + g_n(z) \int \pi_0(x)\, k(x,z)\, dx]$$

Asymptotically this reduces to

$$g(z) = \frac{\pi_1(z)\, r(z)}{1 - \int \pi_0(x)\, k(x,z)\, dx} = \left(\frac{\pi_1(z)}{\int \pi_1(x)\, k(x,z)\, dx} \right) r(z)$$

which we abbreviate by introducing $H(z)$ as follows:

(55) $$g(z) = H(z)\, r(z) .$$

From (55) and (48) the asymptotic expressions for $r(x)$ can be obtained; multiplying (55) by $k(x,z)$, integrating over z, and using (48) we have

(56) $$r(x) = \int k(x,z)\, H(z)\, r(z)\, dz .$$

If we take for $k(x,z)$ and $\pi_1(x)$ the functions stated previously in (21) and (23), respectively, (56) may be solved explicitly for $r(x)$. The result of this solution is

(57) $$r(x) = \frac{1}{2\pi}\left(1 + \frac{\sin \alpha}{2\alpha} \cos x \right)$$

which has a variance of

(58) $$\sigma_1^2 = \frac{\pi^2}{3} - \frac{\sin \alpha}{\alpha} .$$

Comparing (58) with the variance expressions of the linear U and S theories given in (34) and (39) indicates at least one significant difference. The minimum variance in the Identity theory occurs as α approaches zero, or, alternatively, the variance increases monotonically as α increases. In the U and S theories, however, the variance has a minimum when α equals 103 and 110 degrees respectively. (See Table 1 for comparative values of the variance.) Thus varying α would appear to be one way of discriminating between the Identity theory and the two linear theories. (It should be noted that although it is possible to formulate the Identity model within the linear theory context, simple closed form expressions do not result for this model and for this reason it was not discussed previously.)

Uniform Theory

The appropriate recursion obtained by summing (52U) with (49) and (53) is

$$(59) \qquad g_{n+1}(z) = \theta\, \pi_1(z)\, r_n(z) + \frac{\theta}{2\pi} \int \pi_0(x)\, r_n(x)\, dx + (1 - \theta)\, g_n(z)$$

which gives asymptotically

$$g(z) = \pi_1(z)\, r(z) + \frac{1}{2\pi} \int \pi_0(x)\, r_n(x)\, dx \ .$$

The asymptotic response density $r(x)$ is then equal to

$$(60) \qquad r(x) = \int k(x,z)\, \pi_1(z)\, r(z)\, dz + \frac{1}{2\pi} \iint k(x,z)\, \pi_0(x')\, r_n(x')\, dz\, dx' ,$$

but due to the symmetry assumption of the k-distribution $k(x,z) = k(z,x)$ so that the integration over z equals one. Thus (60) becomes

$$(61) \qquad r(x) = \int k(x,z)\, \pi_1(z)\, r(z)\, dz + \frac{1}{2\pi} \int \pi_0(x')\, r(x')\, dx'$$

$$= \int k(x,z)\, \pi_1(z)\, r(z)\, dz + \frac{1}{2\pi} - \int \pi_1(x')\, r(x')\, dx'$$

which clearly agrees with (25). Thus the two uniform theories lead to the same asymptotic response distribution.

Symmetric Theory

To obtain the recursion in $g_n(z)$ for the symmetric model, we let $\gamma = \pi$ in (52S) and combining (52S) with (49) and (53) we obtain

$$(62) \qquad g_{n+1}(z) = (1 - \theta)\, g_n(z) + \theta[\pi_1(z)\, r_n(z) + r_n(z - \pi)\, \pi_0(z - \pi)].$$

At asymptote

$$(63) \qquad g(z) = \pi_1(z)\, r(z) + \pi_0(z - \pi)\, r(z - \pi)\ ,$$

or in terms of $r(x)$

$$(64) \qquad r(x) = \int_{-\pi}^{\pi} k(x,z)\, \pi_1(z)\, r(z)\, dz + \int_{-\pi}^{\pi} k(x,z)\, \pi_0(z - \pi)\, r(z - \pi)\, dz\ .$$

Equation (64) is in fact precisely equivalent to the asymptotic response density of the linear-symmetric theory given in (37). To show this more expicitly we perform a change of variable in the second integral of (64). Let $(z - \pi) = z'$; the second integral then becomes

$$(65) \qquad \int_{-2\pi}^{0} k(x,\ z' + \pi)\, \pi_0(z')\, r(z')\, dz'$$

but since the functions $k(x,\ z' + \pi)$, $\pi_0(z')$, and $r(z')$ are periodic (with period equal to 2π) we may change the limits of integration back to $(-\pi,\ \pi)$. Thus identifying $k_h(x,\ x_n)$ with $k(x,z)$ the equivalence between (64) and (37) is explicitly realized. Thus the two symmetric models give rise to the same asymptotic response distribution. This is not to suggest, however, that the two theories are identical.

Sequential Characteristics

The sequential statistic $r_{n+1}(x|Y_{0,n})$ considered previously will be discussed here as well as for the stimulus sampling theories. First we consider $g_{n+1}(z|Y_{0,n})$ defined by

(66) $$g_{n+1}(z|Y_{0,n}) = \frac{j_{n+1}(z, Y_{0,n})}{P(Y_{0,n})} .$$

The numerator of (66) for the identity theory is obtained by summing (51) and (52I); thus from (66)

(67) $$g_{n+1}(z|Y_{0,n}) = \frac{(1-\theta)\, g_n(z) \int k(x,z)\, \pi_0(x)\, dx + \theta g_n(z) \int k(x,z)\, \pi_0(x)\, dx}{\int \pi_0(x)\, r_n(x)\, dx}$$

which is, of course, equal to

(68) $$g_{n+1}(z|Y_{0,n}) = \frac{g_n(z) \int k(x,z)\, \pi_0(x)\, dx}{\int \pi_0(x)\, r_n(x)\, dx}$$

indicating that this statistic is independent of θ. The conditional response density $r_{n+1}(x|Y_{0,n})$ is obtained directly from (68) by the usual procedure of first multiplying both sides by $k(x,z)$ and then integrating over z:

(69) $$r_{n+1}(x|Y_{0,n}) = \frac{r(x) - \int k(x,z)\, \pi_1(z)\, r_n(z)\, dz}{\int \pi_0(x)\, r_n(x)\, dx} .$$

Using the functions for $r(x)$, $k(x,z)$ and $\pi_1(z)$ given previously in (57), (21), and (23), respectively, we obtain the asymptotic conditional response density $r(x|Y_{0,n})$:

$$(70) \qquad r(x|Y_{0,n}) = \frac{1}{2\pi}\left[1 + \left(\frac{(2\pi - 3\alpha)\sin\alpha}{4\alpha\pi - 4\alpha^2 - \sin^2\alpha}\right)\cos x\right] .$$

The expression in (70), it will be noted, does not agree with either of the two previous expressions for this statistic (see (46) and (47)).

For the U and S theories, $g_{n+1}(z|Y_{0,n})$ is obtained by summing (51) with (52U) and (51) with (52S), respectively. In the former case

$$(71) \qquad g_{n+1}(z|Y_{0,n}) = \frac{(1-\theta)\, g_n(z)\int k(x,z)\,\pi_0(x)\,dx + \frac{\theta}{2\pi}\int \pi_0(x)\,r_n(x)\,dx}{\int \pi_0(x)\,r_n(x)\,dx}$$

and for $\theta = 1$

$$(72) \qquad g_{n+1}(z|Y_{0,n}) = \frac{1}{2\pi} ,$$

indicating that

$$r_{n+1}(x|Y_{0,n}) = \frac{1}{2\pi} .$$

Thus the uniform stimulus sampling theory agrees with the corresponding implications from the corresponding linear theory. In the S-theory,

$$(73) \qquad g_{n+1}(z|Y_{0,n}) = \frac{(1-\theta)\, g_n(z)\int k(x,z)\,\pi_0(x)\,dx + \theta r_n(z-\gamma)\,\pi_0(z-\gamma)}{\int \pi_0(x)\,r_n(x)\,dx}$$

which, unlike (68), depends on θ. If $\theta = 1$,

$$g_{n+1}(z|Y_{0,n}) = \frac{\pi_0(z-\gamma)\, r_n(z-\gamma)}{\int \pi_0(x)\, r_n(x)\, dx} \tag{74}$$

giving for the response density

$$r_{n+1}(x|Y_{0,n}) = \frac{\int k(x, z+\gamma)\, r_n(z)\, \pi_0(z)\, dz}{\int \pi_0(x)\, r_n(x)\, dx} . \tag{75}$$

Since equation (75) and (45) are equivalent when $k_m(x, x') = k(x, x' + \gamma)$, the two symmetric theories are seen to imply the same expressions for $r_{n+1}(x|Y_{0,n})$ for the special case $\theta = 1$.

It should be pointed out perhaps that (71) and (73) can be used to estimate θ for the symmetric and uniform theories so that definite predictions can be made for the sequential characteristics. Expressions such as $r_{n+1}(x|Y_{1,n})$ can easily be obtained in these theories and in all cases they show a dependence on θ.

Extensions

The ideas developed in this paper may be directly extended to other continua than the circumference of a circle. More importantly, the non-determinate conditions of reinforcement may be varied in several directions without disturbing the main lines of the mathematical arguments leading to the mean asymptotic response distributions or some of the simpler sequential statistics. For instance, it is clear how to modify the theory of the basic target experiment described at the beginning in order to predict response behavior when the subject is shown the exact position of the target on those trials on which he misses. Another possibility of some interest is to show the subject the target's exact position on a certain proportion of the trials independently of the correctness or incorrectness of his response.

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