A COMPARISON OF THREE MODELS FOR A HUMPHREYS-TYPE CONDITIONING SITUATION

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TECHNICAL REPORT NO. 5

NOVEMBER 20, 1956

PREPARED UNDER CONTRACT Nonr 225(17)

(NR 171-034)

FOR

OFFICE OF NAVAL RESEARCH

APPLIED MATHEMATICS AND STATISTICS LABORATORY

STANFORD UNIVERSITY

STANFORD, CALIFORNIA

A COMPARISON OF THREE MODELS FOR A HUMPHREYS-TYPE CONDITIONING SITUATION[1]

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Richard C. Atkinson[2]

Summary.

Three models for a Humphreys-type conditioning situation are presented. In model I experimental trials are viewed as discrete units, and the possible influence of trace stimuli on behavior is not considered. Models II and III are members of a class of representations which incorporates a concept of trace stimuli as determining components of subsequent behavior. Functions expressing the expected probabilities of responses are derived and predictions for the three models compared.

1. Introduction.

The purpose of this paper is to provide an analysis of a Humphreys-type conditioning situation in terms of statistical learning theory [3,4,6]. We consider an experimental situation in which each trial begins with the presentation of a signal. Following the signal, one or the other of two reinforcing events, E_1 or E_2, occurs; the probability of E_1 and E_2 during a given series being π and $(1-\pi)$ respectively. The subject is instructed to predict on each trial which event, E_1 or E_2, will occur. The behaviors available to the subject are categorized into two classes, A_1 and A_2; an A_1 response is a prediction by the subject that E_1 will occur, and an A_2 response is a

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[1] This research was supported by the Behavioral Sciences Division of the Ford Foundation and in part by the Office of Naval Research under Contract NR 171-034.

[2] The author wishes to thank Professor P. Suppes for suggestions in carrying out this research.

prediction that E_2 will occur.

In analyzing the situation the experimental psychologist is primarily interested in two questions: (a) what is the relation between π and the asymptotic probability of an A_1 response and (b) what is the relation between π and the rate of approach to the asymptote.

2. Model I. Several investigators [1,2,5,7] have provided the following interpretation of the situation in terms of statistical learning theory. They suggest that the stimulus governing the subjects response on each trial is the signal. The signal is conceptualized as a population, S_c, of stimulus elements which is sampled by the subject on each presentation of the signal; the probability of any given element being sampled is θ. By association principles [4] an element sampled from S_c on a trial will become conditioned to response A_1 if an E_1 event occurs and to response A_2 if an E_2 event occurs. The probability of an A_1 response at the end of trial n is defined in the model as the proportion of elements in S_c that are conditioned to A_1, and similarly for the probability of an A_2.[3/]

We can then define the probability, $p(n)$, that a given element in S_c is conditioned to A_1 at the start of trial n as

(1) $\quad p(n) = (1-\theta)p(n-1) + \theta \quad$ if an E_1 occurred on trial $n-1$,

or

$\quad p(n) = (1-\theta)p(n-1) \quad$ if an E_2 occurred on trial $n-1$.

3/ The reader is referred to Estes and Burke [4] for a statement of the rationale underlying these assumptions.

This leads to an expected difference equation

$$p(n) = (1-\theta)p(n-1) + \theta\pi, \tag{2}$$

whose solution is

$$p(n) = \pi - [\pi - p(0)](1-\theta)^n, \tag{3}$$

where $p(0)$ is the probability that the given element is conditioned to an A_1 response at the start of the first trial.

The mean value of $p(n)$ over all elements in S_c is the expected proportion of elements conditioned to A_1. We have assumed that θ is the same for all elements in S_c, and may therefore interpret $p(n)$ as the probability of an A_1 response at the start of trial n.

By inspection of equation (3) we see model I predicts that (a) the probability of an A_1 response approaches π as n becomes large, and (b) the rate of approach[4] is independent of π.

In the remaining part of this paper we develop alternative formalizations of the stimulus governing the subject's response and investigate the relationships between these models and the above model.

3. Model II. We assume that the stimulus governing the elicitation of a response on each trial is a compound of both (a) the signal stimulus and (b) the reinforcing stimulus of the previous trial.

Let S_c represent the set of stimulus elements associated with the

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[4] Rate of approach, in this paper, refers to the term raised to the power n. For example in equation (3), the term $(1-\theta)$.

signal and S_i the set associated with the occurrence of E_i (i=1,2); assume the three sets are pairwise disjoint. The sampling parameter associated with S_c is θ', with S_1 is θ_1, and with S_2 is θ_2. For most experimental arrangements it is natural to assume $\theta_1 = \theta_2$; hence, to simplify notation, we let $\theta_1 = \theta_2 = \theta$.

Then on trial n the stimulus governing the probability of response is composed of (a) samples from S_c and S_1 if E_1 occurred on trial n-1 and (b) samples from S_c and S_2 if E_2 occurred on trial n-1.

We define the following probabilities.

$p_c(n)$: probability that a given element in S_c is conditioned to A_1 at the start of trial n.

$p_1(n)$: probability that a given element in S_1 is conditioned to A_1 at the start of trial n.

$p_2(n)$: probability that a given element in S_2 is conditioned to A_1 at the start of trial n.

By the same development employed in model I,

$$p_c(n) = \pi - [\pi - p_c(0)](1-\theta')^n . \tag{4}$$

For $p_1(n)$, however, we have a probability π on each trial that S_1 is available for sampling and, in addition, a probability θ that a given element is sampled. That is, on any trial n there is a probability $\theta\pi$ that an element in S_1 is sampled. Hence

(5) $$p_1(n) = (1-\theta\pi)p_1(n-1) + \theta\pi \quad \text{if an } E_1 \text{ occurs on trial } n-1 ,$$

or

$$p_1(n) = (1-\theta\pi)p_1(n-1) \quad \text{if an } E_2 \text{ occurs on trial } n-1 .$$

The expected difference equation is then

(6) $$p_1(n) = (1-\theta\pi)p_1(n-1) + \theta\pi^2 .$$

A similar argument leads to the following expression for $p_2(n)$.

(7) $$p_2(n) = [1-\theta(1-\pi)]p_2(n-1) + \theta(1-\pi)\pi .$$

Solving equations (6) and (7) we obtain

(8) $$p_1(n) = \pi - [\pi-p_1(1)][1-\theta\pi]^{n-1}$$

(9) $$p_2(n) = \pi - [\pi-p_2(1)][1-\theta+\theta\pi]^{n-1}$$

where $p_1(1)$ and $p_2(1)$ represent the probability that a given element is conditioned to A_1 at the start of the second trial.

Next define $p_i[n|E_i]$ as the probability that an element in S_i is conditioned to the A_1 response at the start of trial n, given that an E_i event occurred on trial $n-1$. By conditional probability considerations

(10) $$p_1[n|E_1] = (1-\pi)p_1(n-1) + \pi[(1-\theta)p_1(n-1) + \theta]$$

and

$$p_2[n|E_2] = \pi\, p_2(n-1) + (1-\pi)[(1-\theta)p_2(n-1)] \quad . \tag{11}$$

One final definition is required before we can write the probability of an A_1 response associated with the compound stimulus S_c and S_i. In the presence of S_c and S_1 the effect of S_c on response probability is α_1 and the effect of S_1 is $(1-\alpha_1)$. Similarly, α_2 is defined for S_c and S_2. Again, in most experimental arrangements, it is natural to assume $\alpha_1 = \alpha_2$ and hence we let $\alpha_1 = \alpha_2 = \alpha$.

We can now write the expected probability of an A_1 response at the start of trial n.

$$p(n) = \alpha\, p_c(n) + (1-\alpha)\left\{\pi\, p_1[n|E_1] + (1-\pi)p_2[n|E_2]\right\} . \tag{12}$$

Substituting equations (8) and (9) into equations (10) and (11) and, in turn, substituting the results into equations (12) yields the following expression.

$$\begin{aligned} p(n) = \pi &+ (1-\alpha)\theta[3\pi^2 - \pi - 2\pi^3] \\ &- \alpha[\pi - p_c(0)](1-\theta')^n \\ &- (1-\alpha)\pi[\pi - p_1(1)](1-\theta\pi)^{n-1} \\ &- (1-\alpha)(1-\pi)[\pi - p_2(1)](1-\theta+\theta\pi)^{n-1} \quad . \end{aligned} \tag{13}$$

The function is defined for $n=1,2,\ldots$. For the first trial $(n=0)$ we

let $p(0) = p_c(0)$.

An inspection of equation (13) indicates that for $\alpha < 1$, $p(n)$ approaches an asymptote above π for $\frac{1}{2} < \pi < 1$ and an asymptote below π for $0 < \pi < \frac{1}{2}$. For $\pi = 0, \frac{1}{2}$, or 1 the asymptote is π. Further, the approach to the asymptote is a function of θ, θ' and π. For $\alpha = 1$, equation (13) reduces to equation (3).

4. Model III. We assume that the stimulus which determines response probability on each trial is a compound of the reinforcing stimuli of the two previous trials. More specifically, there are four stimuli, one of which is present on each trial, that determine response probability. We define the following four pairwise disjoint sets of stimulus elements.

S_{ij}: set available for sampling on trial n given that an E_i reinforcing event occurred on trial $n-2$ and an E_j reinforcing event occurred on trial $n-1$, where $i=1,2$ and $j=1,2$.

Again we assume the sampling constants associated with the four sets are equal and denoted by θ.

Next define $p_{ij}(n)$ as the probability that a given element in set S_{ij} is conditioned to the A_1 response at the start of trial n.

By considerations similar to those for equation (5) we obtain for an element in S_{11} a probability π^2 that the set S_{11} is available for sampling on a given trial and, hence, a probability $\theta\pi^2$ that a given element S_{11} is sampled on the trial. Therefore

$$(14) \qquad p_{11}(n) = (1-\theta\pi^2)p_{11}(n-1) + \theta\pi^2 \quad \text{if } E_1 \text{ occurred on trial } n-1$$

or

$$p_{11}(n) = (1-\theta\pi^2)p_{11}(n-1) \quad \text{if} \quad E_2 \quad \text{occurred on trial} \quad n-1.$$

This leads to the expected difference expression

$$p_{11}(n) = (1-\theta\pi^2)p_{11}(n-1) + \theta\pi^3 \ . \tag{15}$$

By identical considerations we obtain

$$p_{12}(n) = [1-\theta\pi(1-\pi)]p_{12}(n-1) + \theta(1-\pi)\pi^2 \tag{16}$$

$$p_{21}(n) = [1-\theta\pi(1-\pi)]p_{21}(n-1) + \theta(1-\pi)\pi^2 \tag{17}$$

$$p_{22}(n) = [1-\theta(1-\pi)^2]p_{22}(n-1) + \theta(1-\pi)^2\pi \ . \tag{18}$$

Next define $p_{ij}[n|E_iE_j]$ as the probability that an element in S_{ij} is conditioned to A_1 at the start of trial n given that an E_i event occurred on trial $n-2$ and E_j on trial $n-1$. By conditional probability considerations

$$p_{11}[n|E_1E_1] = \pi^2[p_{11}(n-2)(1-\theta)^2 + \theta(1-\theta) + \theta] + [1-\pi^2]p_{11}(n-2) \ , \tag{19}$$

$$p_{12}[n|E_1E_2] = \pi(1-\pi)[p_{12}(n-2)(1-\theta) + \theta] + [1-\pi(1-\pi)]p_{12}(n-2) \ , \tag{20}$$

$$p_{21}[n|E_2E_1] = \pi(1-\pi)[p_{21}(n-2)(1-\theta)] + [1-\pi(1-\pi)]p_{21}(n-2) \ , \tag{21}$$

$$p_{22}[n|E_2E_2] = (1-\pi)^2[p_{22}(n-2)(1-\theta)^2] + [1-(1-\pi)^2]p_{22}(n-2) \ . \tag{22}$$

We can now define the expected probability of an A_1 response on trial n as

$$(23) \qquad p(n) = \pi^2 p_{11}[n|E_1E_1] + \pi(1-\pi)p_{12}[n|E_1E_2] + \pi(1-\pi)p_{21}[n|E_2E_1] + (1-\pi)^2 p_{22}[n|E_2E_2] .$$

Solving recursive expressions (15)-(18), substituting the results in equations (19)-(22), and in turn substituting these results in equation (23) we obtain for the probability of an A_1 response at the start of trial n

$$(24) \qquad p(n) = \pi + (2\theta - \theta^2)[\pi^4(1-\pi) - \pi(1-\pi)^4] - \theta[(1-\pi)^2\pi^2(2\pi-1)] - \pi^2[1 - \pi^2(2\theta - \theta^2)]\gamma_{11}\beta_{11}^{n-2} - \pi(1-\pi)[1 - \theta\pi(1-\pi)][\gamma_{12}\beta_{12}^{n-2} + \gamma_{21}\beta_{21}^{n-2}] - (1-\pi)^2[1 - (1-\pi)^2(2\theta - \theta^2)]\gamma_{22}\beta_{22}^{n-2}$$

where $\gamma_{ij} = \pi - p_{ij}(2)$, $\beta_{11} = 1 - \theta\pi^2$, $\beta_{22} = 1 - \theta(1-\pi)^2$, and $\beta_{12} = \beta_{21} = 1 - \theta\pi(1-\pi)$. The function is defined for $n=2,3,\ldots$. In dealing with most experimental situations where no initial preference exists between A_1 and A_2 it would be reasonable to assume $p(0) = \frac{1}{2}$ and $p(1) = (1-\theta)\frac{1}{2} + \theta\pi$.

5. Comparison of Model I and Model III. In this section we are concerned with a comparison between model I and III. But it should be noted that for all comparisons the result obtained by model II, for any α, will be bounded by the results of models I and III.

Let $p_I(n)$ be the probability of an A_1 response defined in equation (3) and $p_{III}(n)$ be the probability of the same response as defined in equation (24). Further, for simplicity let $p_I(0) = p_{III}(0) = \frac{1}{2}$ and, since $p_{III}(1)$ is not defined, let $p_I(1) = p_{III}(1)$.

An inspection of equations (3) and (24) indicates that the asymptotic values for model I and model III are equal for $\pi = 0, \frac{1}{2}$, or 1. In the interval $0 < \pi < \frac{1}{2}$, $p_I(\infty) > p_{III}(\infty)$ while for $\frac{1}{2} < \pi < 1$, $p_I(\infty) < p_{III}(\infty)$.

Next, define the functions

$$\chi_I(N,\pi) = N\pi - \sum_{i=0}^{N-1} p_I(i) , \tag{25}$$

and

$$\chi_{III}(N,\pi) = N\pi - \sum_{i=0}^{N-1} p_{II}(i) . \tag{26}$$

For $\pi = 1$

$$\chi_I(N,1) = \chi_{III}(N,1)$$

and

$$\lim_{N \to \infty} \chi(N,1) = \frac{1}{2\theta} . \tag{27}$$

Using the value of θ obtained in equation (27) we can compute $\chi_I(N,\pi)$ and $\chi_{III}(N,\pi)$ for any value of π. $\chi_I(N,\pi) = \chi_{III}(N,\pi)$; for $\pi = 0, \frac{1}{2}$, or 1; $\chi_I(N,\pi) < \chi_{III}(N,\pi)$, for all other values of π. Stated differently, the rate of approach to the asymptote for $\pi = 0$ or 1 is identical for models I and III, but for other values of π, the rate predicted by model I is greater than the prediction by model III.

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