THE AXIOMATIC METHOD IN HIGH-SCHOOL MATHEMATICS

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<u>Introduction</u>. The importance of the axiomatic method in modern mathematics scarcely needs a general defense. Its widespread use in many parts of mathematics and its long history of importance in the mathematics of earlier centuries provide clear evidence that it will continue to be of importance in the foreseeable future of mathematics. On the other hand, the role of the axiomatic method in high-school mathematics is not as universally accepted, even though it has had a place in the teaching of school geometry throughout the history of western culture since the time of Eudoxus. In the last decade the relevance of the axiomatic method even to the teaching of high-school geometry has been challenged in some quarters.

The point of view that I want to present here is to make as vigorous a defense as I can of the importance of teaching the axiomatic method in high-school mathematics. I realize full well that it is not sufficient as an argument to point to the importance of the axiomatic method already in the teaching of university mathematics and in mathematical research.

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There are three general arguments I would like to advance for emphasizing the axiomatic approach to mathematics in the high-school curriculum. In the first place the axiomatic approach provides an important method of making the mathematics taught more elementary and the subject more restricted from the standpoint of what the student has to learn and encompass. An excellent example is provided by the real numbers. We now expect students to know a good deal about the real number system by the time they have finished high-school mathematics, at least the students who are college bound and who take a full high-school mathematics curriculum. If the properties of the real number system are taught from an axiomatic standpoint, the student is given a restricted list of properties that are fundamental and from which all others can be derived. Moreover most of the axioms that are used express elementary properties that generalize in a natural way what the student has learned in earlier years. The axiomatic approach also assures the student that the properties he must know are restricted to those expressed by the axioms and the theorems that follow from the axioms. It is my own belief that it is all too easy for us confidently to assume that students are clear about the properties of the real numbers and have a good intuitive feeling for what the real numbers are. This assumption, which I feel is unwarranted, is relatively crucial to those approaches to secondary-school geometry that lean heavily on the properties of the real numbers.

Perhaps the best persepctive in which to consider the axiomatic approach to the real number system is to ask what are the alternatives. It is doubtful that many people feel it would be wiser to go through the full mathematical construction of the real numbers via Dedekind cuts or Cauchy sequences of the rational numbers, and moreover it is doubtful that they would want to construct the rational numbers themselves as certain equivalence classes of ordered pairs of integers. An historically important alternative approach has been through the geometrical theory of real magnitudes, but it is doubtful that anyone will want the real numbers to be constructed out of geometrical entities. In any case either of these prospective approaches is certainly less elementary than the axiomatic algebraic approach. I suppose that a fourth alternative is simply to leave the whole matter up in the air and to develop in a higgledy-piggledy fashion the properties needed, but I am skeptical that students will have the right sort of confidence and clarity about the properties of the real numbers when this approach is taken seriously.

My second argument centers around the importance of developing intuitions for finding and giving mathematical proofs. It is a common complaint about beginning graduate students in mathematics in the United States that one of their worst defects is their inability to write a coherent mathematical proof. A fortiori, this is even truer of undergraduate students of mathematics. And the reasons for this deficiency are not hard to find. Both at the school and university level explicit training in the writing of mathematical proofs and the explicit consideration of heuristic methods for finding proofs are woefully lacking in the curriculum. Systematic pursuit of the axiomatic method in high-school mathematics provides perhaps the best opportunity for training students at an early stage in the finding and writing of mathematical proofs. It is all too easy to assume that as students develop an intuition for geometrical facts, for example, they will almost automatically be able to produce coherent proofs. Put another way, what I am saying is that I consider it just as necessary to train the intuition for finding and writing mathematical proofs as to teach intuitive knowledge of geometry or the real number system. I am contending that training in the finding and writing of proofs must be given as much explicit attention and should begin as early as other parts of the mathematical training of students; it is in the context of the elementary mathematics taught in high school that the student can first learn to work in a natural and easy way with axioms and the proofs of theorems that follow rigorously from the axioms.

My third argument for the use of the axiomatic method in high-school mathematics centers around the increasing importance of learning how to think in a mathematical fashion as the total body of mathematics itself increases so rapidly. It is becoming clear that it is reasonably hopeless to expect students at any stage to master any substantial portion of extant mathematics. We can, of course, agree on the greater importance of certain parts of mathematics, but still a good case can be made that perhaps the best thing we can do for our students is to begin to teach them to think mathematically as effectively as possible. It has been my own experience that mathematicians discussing curriculum are uneasy in any attempt to characterize what they consider to be the essential nature of mathematical thinking. As mathematicians they are much more accustomed to thinking about mathematical objects and proving facts about these objects. In the same way, it is much easier to get a clear and sophisticated statement from a mathematician about why a given proof is correct or another proof contains an error than it is to get a subtle or relatively elaborate evaluation of the worth of one kind of heuristic reasoning versus another. Using an axiomatic approach in the teaching of mathematics provides a superb opportunity for more explicit emphasis on how proofs are found, and what heuristic ideas are central to their discovery.

I would also like to urge the viewpoint that there is no fundamental conflict between the axiomatic method as pursued in pure mathematics and the development of skills for solving problems in applied mathematics. It has become all too fashionable at the present time to emphasize a conflict between pure and applied mathematics, and to be for the one and against the other in terms of what is to be emphasized in the mathematical training of students. From a psychological standpoint there is a very close affinity between the correct and complete statement of the mathematical conditions that characterize a problem in applied mathematics and the statement of axioms in pure mathematics. In both cases the aim is from a mathematical standpoint to make the problem at hand a tub on its own bottom, so to speak, without dependence in implicit and ill-understood ways on other parts of mathematics or on other physical side conditions. An experience that I would claim is psychologically identical to isolating the mathematical features of an applied problem is that of finding axioms for some part of pure mathematics. A weakness of our teaching of the axiomatic method is that we too seldom confront our students with the problem of formulating axioms, as opposed to deriving consequences from a clearly stated, teacher-provided set of axioms. In my own judgment the present teaching of axiomatic mathematics at the school or university level is more deficient on this point than any other.

To give these general remarks a greater sense of definiteness, I now turn to some more constructive and particular ideas about the axiomatic method under the headings of logic, algebra, geometry and calculus.

Logic. As part of training in the axiomatic method in school mathematics, I would not advocate an excessive emphasis on logic as a self-contained discipline. For example, I do not really agree with those mathematicians who feel that logic should be studied in the form of Boolean algebra as an autonomous discipline early in the mathematical training of students. What I do feel is important is that students be taught in an explicit fashion classical rules of logical inference, learn how to use these rules in deriving theorems from given axioms, and to come to feel as much at home with simple principles of inference like modus ponendo ponens as they do with elementary algorithms of arithmetic. I hasten to add that these classical and ubiquitous rules of inference need not be taught in symbolic form, nor do students need to be trained to write formal proofs in the sense of mathematical logic. What I have in mind is that the student should be able to recognize without second thought the correctness of the inference:

> If this figure is a square, then this figure is a quadrilateral.
> This figure is a square.

Therefore, this figure is a quadrilateral.

And also to recognize the fallacious character of the inference:

If this figure is a square, then this figure is a quadrilateral.
This figure is a quadrilateral.
Therefore, this figure is a square. (Fallacious)

The classical forms of sentential inference present no problem, and are already covered in many of the modern textbooks on high-school geometry. The real pedagogical problem centers around the making of valid inferences involving quantifiers. In this respect it seems to me that the best approach is the classical one of divide and conquer. What I mean by this is that students should first be introduced to substitution for individual variables where the only quantifiers implicitly understood are universal quantifiers standing at the beginning of a sentence and whose scopes are the remainder of the sentence. No other universal quantifiers and no existential quantifiers of any sort should be considered at this stage. For this restricted use of quantifiers, essentially only a simple rule of substitution of terms for variables and a correspondingly simple rule of generalization is required. In the paragraphs below I try to indicate how far this sort of logic can carry us in the elementary treatment of the algebra of real numbers and of vector geometry, without requiring the introduction of existential quantifiers, and the subtle problems of inference that accompany these quantifiers. Only at a late stage in high-school mathematics would I recommend that inferences involving existential quantifiers be explicitly introduced, and then only sparingly.

The role I see for logic in teaching of the axiomatic method in high-school mathematics should be clear. Without training in the proving of theorems the development of the axiomatic method is a sterile enterprise. It is important and essential that students learn how to make inferences from axioms in order to comprehend the power of the axiomatic method. To be able to make such inferences, they should be given training in the standard forms of inference that they may use in learning to think out and write down an acceptable mathematical proof. From years of grading mathematical proofs given on examinations at the university level, I am firmly convinced, as I have already indicated, that the ability to write a coherent mathematical proof does not develop naturally even at the most elementary levels and must be a subject of explicit training.

Algebra. The initial framework of logic described above, it is suggested, should be deliberately restricted to quantifier-free sentences in order to avoid the troublesome and subtle matter of handling existential quantifiers or universal quantifiers with restricted scope. The student is already familiar with this logic, for it corresponds rather closely to the elementary arithmetic and algebra he has had prior to entering high school. The bulk of the algebra he has learned can now be codified in an elementary axiomatic fashion by deriving the consequences of the axioms for a Euclidean field, that is, an ordered field in which every non-negative element is a square. We may avoid all existential quantifiers by replacing the three existential axioms.

$$(\forall x)(\exists y)(x + y = 0) ,$$

$$(\forall x)(\forall y)[\underline{\text{if}} \quad y \neq 0 \quad \underline{\text{then}} \quad (\exists z)(x = y \cdot z)] ,$$

$$(\forall x)(\underline{\text{if}} \quad 0 < x \quad \underline{\text{then}} \quad (\exists y)(x = y \cdot y)$$

by the following three axioms which introduce the operations of subtraction, division and taking the square root of a positive number.

$$x - y = z \quad \underline{\text{if}}\ \underline{\text{and}}\ \underline{\text{only}}\ \underline{\text{if}} \quad x = y + z .$$

$$\underline{\text{If}} \quad y \neq 0 \quad \underline{\text{then}} \quad x \div y = z \quad \underline{\text{if}}\ \underline{\text{and}}\ \underline{\text{only}}\ \underline{\text{if}} \quad x = y \cdot z .$$

$$\underline{\text{If}} \quad 0 < x \quad \underline{\text{then}}\ \sqrt{x} = y \quad \underline{\text{if}}\ \underline{\text{and}}\ \underline{\text{only}}\ \underline{\text{if}} \quad x = y \cdot y .$$

In these terms then the elementary algebra taught in high school should mainly center around the consequences of the axioms that define a Euclidean field. In view of the three axioms just stated, which are introduced to eliminate existential quantifiers, the axioms as stated here for a Euclidean field use the operation symbols for addition, multiplication, subtraction, division, and taking the square root, the relation symbol " $<$" and the individual constants "0" and "1". The beauty of these axioms is that their intuitive content should be familiar to the students. Perhaps the only idiom that needs some explicit new discussion is the use of " if and only if" in the three axioms just mentioned. The full set of axioms is the following:

(1) $$x + y = y + x$$

(2) $$x \cdot y = y \cdot x$$

(3) $$(x + y) + z = x + (y + z)$$

(4) $$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

(5) $x \cdot (y+z) = (x \cdot z) + (x \cdot z)$

(6) $x + 0 = x$

(7) $x \cdot 1 = x$

(8) $x - y = z$ *if and only if* $x = y + z$

(9) *If* $y \neq 0$ *then* $x \div y = z$ *if and only if* $x = y \cdot z$.

(10) *If* $x < y$ *then it is not the case that* $y < x$.

(11) *If* $x < y$ *and* $y < z$ *then* $x < z$.

(12) *If* $x \neq y$ *then* $x < y$ *or* $y < x$.

(13) *If* $y < z$ *then* $x + y < x + z$.

(14) *If* $0 < x$ *and* $y < z$ *then* $x \cdot y < x \cdot z$.

(15) *If* $0 < x$ *then* $\sqrt{x} = y$ *if and only if* $x = y \cdot y$.

(16) $0 \neq 1$.

At a later stage and toward the end of high-school mathematics for those who are taking a full program, it will be appropriate to go on to the concept of a real closed field, that is, a field that is Euclidean and is such that every polynomial of an odd degree with coefficients in the field has a zero in the field. (The choice of real closed fields as a terminal algebraic concept rests on the fact that every real closed field is elementarily equivalent with the field of real numbers; by this I mean that every first-order sentence which holds in one of these two fields holds in the other, and by 'first-order' is meant sentences whose variables range only over elements of the field and not over sets of elements.)

Geometry. The appropriate axiomatic approach to elementary geometry is, as everyone knows, a much more controversial subject.

If a vector-space approach is used, then it is possible to use the quantifier-free methods just described for algebra, and this has the important advantage of continuing to keep the structure of proofs simple. It also means that proofs can be a natural extension of the techniques already learned in algebra. Moreover, in line with the earlier remarks on algebra, I would propose vector spaces over Euclidean fields as the proper elementary objects of the theory. The axiomatic approach can take proper advantage of the fact that the vectors form an abelian group under addition just as the real numbers or elements of a Euclidean field do, and therefore all elementary properties are shared. Elementary theorems about addition of vectors will have already been proved as elementary theorems about addition of numbers. To this vector-space structure may be added the concept of the inner product of two vectors to permit the introduction of concepts of distance and perpendicularity. The quantifier-free axioms on the inner product are just the following three, where α and β are real numbers, and x, y and z are vectors.

$$\text{If } x \neq 0 \text{ then } x \cdot x > 0 ,$$

$$x \cdot y = y \cdot x ,$$

$$(\alpha x + \beta y) \cdot z = \alpha(x \cdot z) + \beta(y \cdot z) .$$

Unfortunately, for the treatment of many geometrical figures and their properties, which we expect our students to know, the purely vector-space approach does not provide a natural framework. For these developments, it is my own conviction that an intrinsic axiomatization that emphasizes the role of geometrical constructions is the most appealing. I realize, however, that there is wide disagreement on this viewpoint.

There are also pedagogical difficulties in providing a strictly axiomatic approach in terms of geometrical constructions. I will not pursue the point further here. It should be mentioned that still a third approach to elementary geometry is in terms of introducing geometrical transformations as well as vectors. It is thoroughly clear from recent discussions that it will be some time before the pedagogically most suitable set of axioms will be hit upon in terms of any of the approaches I have mentioned, but I would like to emphasize the importance of quantifier-free methods if we expect our students to become adept at finding and writing correct proofs. The logical complexities of most axiomatic approaches to geometry at the high-school level make it difficult for students to acquire a clear and sure-footed understanding of what mathematical arguments are all about.

Calculus. Space does not permit many comments on how the axiomatic approach may be applied to the teaching of calculus in high school, but the main thrust of what I want to say can be easily conjectured from what I have already said about algebra and geometry. An axiomatic approach in terms of 'epsilon-delta' concepts and proofs does not seem appropriate. What does seem practical is an axiomatic algebraic approach to the calculus of elementary functions combined with considerable stress on the intuitive geometric and physical meaning of the derivative and integral of an elementary function. Moreover, the student can explictly check the axioms by computing the areas of rectangles and triangles or the properties of rectilinear motion.

Again, I emphasize that a quantifier-free approach permits an easy but rigorous development of the elementary parts of the calculus, and the student can quickly be led to have a feel for the power of the calculus in solving empirically meaningful problems.*

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*For those interested in quantifier-free arithmetic, an excellent survey is to be found in J. C. Shepherdson, "Non-standard models for fragments of number theory," The Theory of Models, edited by J. W. Addison, L. Henkin and A. Tarski, North-Holland Publishing Company, Amsterdam, 1965. One beautiful result is due to J. R. Shoenfield, "Open sentences and the induction axiom," Journal of Symbolic Logic, vol. 23 (1958), pp. 7-12. He proves that a system of nine axioms based on the successor ', predecessor P and addition + operations, and constant 0 is not augmented in deductive power by the addition of the inductive axiom for sentences without quantifiers. The nine axioms are just these.

(1) $x' \neq 0$

(2) $PQ = 0$

(3) $Px' = x$

(4) $x + 0 = x$

(5) $x + y' = (x + y)'$

(6) If $x \neq 0$ then $x = (Px)'$

(7) $x + y = y + x$

(8) $(x + y) + z = x + (y + z)$

(9) If $x + y = x + z$ then $y = z$.