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CHAPTER

6

MEASUREMENT, EMPIRICAL MEANINGFULNESS, AND THREE-VALUED LOGIC[1]

Patrick Suppes

ASSOCIATE PROFESSOR OF PHILOSOPHY
STANFORD UNIVERSITY

INTRODUCTION

The predominant current opinion appears to be that it is scarcely possible to set up criteria of empirical meaningfulness for individual statements. What is required, it is said, is an analysis of theories taken as a whole. There is even some skepticism regarding this, and it has been romantically suggested that the entire fabric of experience and language must be considered and taken into account in any construction of general categories of meaning or analyticity. What I have to say makes no contribution to the attempt to find a general criterion of meaning applicable to arbitrary statements. Rather I am concerned to exemplify a general method that will yield specific positive criteria for specific branches of science.

A brief analysis of two simple examples will indicate the sort of thing I have in mind. Consider the statement:

[1] This research was supported in part by the Group Psychology Branch of the Office of Naval Research. I am indebted to G. Kreisel for several helpful comments on an earlier draft of this article. [*Editorial Note:* This paper was separately invited and was not delivered at the Symposium on Measurement.]

(i) *The mass of the sun is greater than* 10^6.

If a physicist were asked if (i) is true or false, he would most likely reply that it depends on what unit of mass is implicitly understood in uttering (i). On the other hand, if we were to ask him about the truth of the sentence:

(ii) *The mass of the sun is at least ten times greater than that of the earth,*

he would, without any reservation about units of measurement, state that (ii) is true, and perhaps even add that its truth is known to every schoolboy. Now my main point is that we may insist that our systematic language of physics (or of any other empirical science) has no hidden references to units of measurement. The numerals occurring in the language are understood to be designating "pure" numbers. An excellent example of a physical treatise written without reference to units is provided by the first two books of Newton's *Principia*. (Units are introduced in the consideration of data in Book III, and occasionally in examples in the earlier books.) Newton avoids any commitment to units of measurement by speaking of one quantity being proportional to another or standing in a certain ratio to it. Thus he formulates his famous second law of motion:

> The change of motion is proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed. (Cajori edition, p. 13.)

Systematic reasons for adopting Newton's viewpoint as the fundamental one are given in later sections. My only concern at the moment is to establish that adoption of this viewpoint does not represent a gross violation of the use of physical concepts and language by physicists. It seems obvious that, in *using* a unitless language, we would not find occasion to use (i), for there would be no conceivable way of establishing its truth or falsity, either by empirical observation or logical argument. In contrast, (ii) would be acceptable. Yet it is difficult to see how to develop a simple and natural syntactical or semantical criterion within, say, a formal language for expressing the results of measurements of mass, which would rule out sentences like (i) and admit sentences like (ii). The central purpose of this paper is to explore some of the possibilities for classifying as meaningless well-formed sentences like (i), or, more exactly, the analogues of (i) in a formalized language. Formalization of a certain portion of the unitless language of physicists is not absolutely necessary for expressing the ideas I want to put forth, but it is essential to a clear working out of details. Moreover, the exact formal construction seems to pose some interesting problems which could scarcely be stated for a natural language. In the final section, the possibility is explored of interpreting this formalized language in terms of a three-valued logic of truth, falsity, and meaninglessness.

INVARIANCE AND MEANINGFULNESS

In connection with any measured property of an object, or set of objects, it may be asked how unique is the number assigned to measure the property. For example, the mass of a pebble may be measured in grams or pounds. The number assigned to measure mass is unique once a unit has been chosen. A more technical way of putting this is that the measurement of mass is unique up to a similarity transformation.[2] The measurement of temperature in °C or °F has different characteristics. Here an origin as well as a unit is arbitrarily chosen: technically speaking, the measurement of temperature is unique up to a linear transformation.[3] Other formally different kinds of measurement are exemplified by (1) the measurement of probability, which is absolutely unique (unique up to the identity transformation), and (2) the ordinal measurement of such physical properties as hardness of minerals, or such psychological properties as intelligence and racial prejudice. Ordinal measurements are commonly said to be unique up to a monotone-increasing transformation.[4]

Use of these different kinds of transformations is basic to the main idea of this paper. An empirical hypothesis, or any statement in fact, which uses numerical quantities is empirically meaningful only if its truth value is invariant under the appropriate transformations of the numerical quantities involved. As an example, suppose a psychologist has an ordinal measure of I.Q., and he thinks that scores $S(a)$ on a cer-

[2] A real-valued function ϕ is a similarity transformation if there is a positive number α such that for every real number x

$$\phi(x) = \alpha x.$$

In transforming from pounds to grams, for instance, the multiplicative factor α is 453.6.

[3] A real-valued function ϕ is a linear transformation if there are numbers α and β with $\alpha > 0$ such that for every number x

$$\phi(x) = \alpha x + \beta.$$

In transforming from Centrigrade to Fahrenheit degrees of temperature, for instance, $\alpha = 9/5$ and $\beta = 32$.

[4] A real-valued function ϕ is a monotone increasing transformation if, for any two numbers x and y, if $x < y$, then $\phi(x) < \phi(y)$. Such transformations are also called *order-preserving*.

tain new test T have ordinal significance in ranking the intellectual ability of people. Suppose further that he is able to obtain the ages $A(a)$ of his subjects. The question then is: Should he regard the following hypothesis as empirically meaningful?

Hypothesis 1. *For any subjects a and b if* $S(a)/A(a) < S(b)/A(b)$, *then* I.Q. $(a) <$ I.Q. (b).

From the standpoint of the invariance characterization of empirical meaning, the answer is negative. To see this, let I.Q. $(a) \geq$ I.Q. (b), let $A(a) = 7$, $A(b) = 12$, $S(a) = 3$, $S(b) = 7$. Make no transformations on the I.Q. data, and make no transformations on the age data. But let ϕ be a monotone-increasing transformation which carries 3 into 6 and 7 into itself. Then we have

$$\tfrac{3}{7} < \tfrac{7}{12},$$

but

$$\tfrac{6}{7} \geq \tfrac{7}{12},$$

and the truth value of Hypothesis 1 is not invariant under ϕ.

The empirically significant thing about the transformation characteristic of a quantity is that it expresses in precise form how unique is the structural isomorphism between the empirical operations used to obtain a given measurement and the corresponding arithmetical operations or relations. If, for example, the empirical operation is simply that of ordering a set of objects according to some characteristic, then the corresponding arithmetical relation is that of less than (or greater than), and any two functions which map the objects into numbers in a manner preserving the empirical ordering are adequate. More exactly, a function f is adequate if, and only if, for any two objects a and b in the set, a stands in the given empirical relation to b if and only if

$$f(a) < f(b).^{5}$$

It is then easy to show that, if f_1 and f_2 are adequate in this sense, then they are related by a monotone-increasing transformation. Only those arithmetical operations and relations which are invariant under monotone-increasing transformations have any empirical significance in this situation.

The key notion referred to in the last sentence is that of invariance. In order to make the notion of invariance or the related notion of meaningfulness completely precise, we can do one of two things: set up an

[5] For simplicity we shall consider here only the arithmetical relation $<$. There is no other reason for excluding $>$.

exact set-theoretical framework for our discussion (e.g., for classical mechanics, see [1]), or formalize a language adequate to express empirical hypotheses and facts involving numerical quantities. Here we shall formalize a simple language for expressing the results of mass measurements. It should be clear that the method of approach is applicable to any other kind of measurement, or combinations thereof.

EMPIRICAL MEANINGFULNESS IN THE LANGUAGE L_M

To avoid many familiar details, we shall use as a basis the formal language of Tarski's monograph [2] enriched by individual variables 'a', 'b', 'c', $\cdots$, 'a_1', 'b_1', 'c_1', $\cdots$, the individual constants: $o_1, \cdots, o_{10}$, which designate ten, not necessarily distinct, physical objects, and the mass term 'm', where '$m(a)$' designates a real number, the mass of a. The values of the individual variables are physical objects. The numerical variables are 'x', 'y', 'z', $\cdots$, 'x_1', 'y_1', 'z_1', $\cdots$. Tarski's numerical constants are: 1, 0, -1. We shall include, for purposes of examples, numerical constants for the positive and negative integers less than 100 in absolute value. The operation signs are those for addition and multiplication. We also include the standard sign for exponentiation with the fixed base 2. A *term* is any arithmetically meaningful expression built up from this notation in the usual manner. (We omit an exact definition.) Thus the following are terms: $m(a)$, $5 \cdot m(a) + 3$, $2 + 1$, $x + 3$, 2^x. Our two relation symbols are the usual sign of equality and the greater than sign. An *atomic formula* is then an expression of the form

$$(\alpha = \beta), \qquad (\alpha > \beta)$$

where α and β are terms with the restriction in the case of $(\alpha > \beta)$ that α and β are both numerical terms, that is, neither α nor β is an individual variable or constant. When no confusion will result, parentheses are omitted. *Formulas* are constructed from atomic formulas by means of sentential connectives and quantifiers. The symbol '$-$' is used for negation; the ampersand '&' for conjunction; the symbol '$\vee$' for disjunction (to be read 'or'); the arrow '$\rightarrow$' for implication (to be read 'if . . . then . . .'); the double arrow '$\leftrightarrow$' for equivalence (to be read 'if and only if'); the reverse '$\exists$' is the existential quantifier; and the upside down '$\forall$' the universal quantifier. Thus the following are formulas: $(\exists x)(m(a) = x)$, $(\exists x)(\exists y)(x > y)$, $0 > x \rightarrow m(b) > x$. We also use the standard symbol '$\neq$' for negating an equality. A formula is a *sentence* if it contains no free variables, that is, every occurrence of a variable is bound by some quantifier.

Sentences are true or false, but unlike the situation in the language of Tarski's monograph [2], the truth or falsity of many sentences in the language L_M constructed here depends on empirical observation and contingent fact. For example, the truth of the sentence:

$$(\exists a)(\forall b)(b \neq a \rightarrow m(a) > 5 \cdot m(b)) \tag{1}$$

is a matter of physics, not arithmetic.

Pursuing now in more detail the remarks in the first section, the intuitive basis for our classification of certain formulas of L_M as empirically meaningless may be brought out by considering the simple sentence:

$$m(o_1) = 4. \tag{2}$$

It must first be emphasized that in the language L_M, the numeral '4' occurring in Sentence 2 designates a "pure" number. There is no convention, explicit or implicit, that '4' stands for '4 g', '4 lb', or the like. It is to be clearly understood that no unit of mass is assumed in the primitive notation of L_M. With this understanding in mind, it is obvious that no experiment with apparatus for determining the masses of physical objects could determine the truth or falsity of Sentence 2. It is equally obvious that no mathematical argument can settle this question. On the other hand, it is clear that sentences like:

$$m(o_1) > m(o_2) \tag{3}$$

or

$$m(o_3) = 5 \cdot m(o_4) \tag{4}$$

which are concerned with numerical relations between the masses of certain objects can be determined as true or false on the basis of experiment without prior determination of a unit of mass.

It seems to me that the use of "pure" numerals in L_M is more fundamental than the use of what we may term "unitized numerals". The justification of this view is that the determination of units and an appreciation of their empirical significance comes *after*, not before, the investigation of questions of invariance and meaningfulness. The distinction between Sentence 2 and the other three Sentences 1, 3, and 4 is that the latter sentences remain true (or false) under any specification of units. In other words, the truth value of these sentences is independent of the arbitrary choice of a unit. Paraphrasing Weyl, we may say:[6] *only the numerical masses of bodies relative to one another have an objective meaning.*

[6] Weyl's original statement is with respect to Galileo's principle of relativity, "Only the motions of bodies (point-masses) relative to one another have an objective meaning." (See [3].)

My claim regarding fundamentals may be supported by an axiomatic, operational analysis of any actual experimental procedure for measuring mass. Most such procedures may be analyzed in terms of three formal notions: the set A of physical objects, a binary operation Q of comparison, and a binary operation $*$ of combination. The formal task is to show that under the intended empirical interpretation the triple $\mathfrak{A} = \langle A, Q, *\rangle$ has such properties that it may be proved that there exists a real-valued function $\mathbf{m}$ defined on A such that for any a and b in A

(i) $$aQb \text{ if and only if } \mathbf{m}(a) \leq \mathbf{m}(b),$$

(ii) $$\mathbf{m}(a * b) = \mathbf{m}(a) + \mathbf{m}(b).$$

The empirically arbitrary character of the choice of a unit is established by showing that the functional composition of any similarity transformation ϕ with the function $\mathbf{m}$ yields a function $\phi \circ \mathbf{m}$ which also satisfies (i) and (ii), where $\circ$ is the operation of functional composition.[7]

We may think of such an operational analysis as supporting the choice of L_M, where the term 'm' of L_M designates a numerical representing function satisfying (i) and (ii). Roughly speaking, because this representing function is only unique up to a similarity transformation, we then expect any sentence to be empirically meaningful in L_M if and only if its truth value is the same when 'm' is replaced by any expression which designates multiplication of the representing function by a positive number. However, there are certain difficulties with deciding exactly how to make this intuitive definition of empirical meaningfulness precise. For example, if the definition applies to any sentences, then we have the somewhat paradoxical result that Sentence 2 and its negation are both empirically meaningless, but their disjunction:

(5) $$m(o_1) = 4 \vee m(o_1) \neq 4$$

is meaningful, since it is always true.

To facilitate our attempts to meet this problem, we first need to introduce the semantical notion of a *model* of L_M. For simplicity in defining the notion of model, and without any loss of generality, we shall from this point on consider L_M as not having any individual constants that designate physical objects.

On this basis, a model $\mathfrak{M}$ of L_M is an ordered triple $\langle \mathfrak{S}, A, \mathbf{m}\rangle$, where

[7] An axiomatic analysis in terms of these ideas may be found in [4]. However, the analysis given in [4] may be criticized on several empirical counts; for example, the set A must be infinite.

(i) $\mathfrak{S}$ is the usual system of real numbers under the operations of addition, multiplication, and exponentiation with the base 2, and the relation less than with the appropriate numbers corresponding to their numerical designations in L_M;[8]

(ii) A is a finite, nonempty set;

(iii) **m** is a function on A which takes positive real numbers as values. The intended interpretation of A is as a set of physical objects whose masses are being determined; the function **m** is meant to be a possible numerical function used to represent experimental results. We assume the semantical notion of *satisfaction* and suppose it to be understood under what conditions a sentence of L_M is said to be *satisfied* in a model $\mathfrak{M}$. Roughly speaking, a sentence S of L_M is satisfied in $\mathfrak{M} = \langle \mathfrak{S}, A, \mathbf{m} \rangle$ if S is true when the purely arithmetical symbols of S are given the usual interpretation in terms of $\mathfrak{S}$, when the individual variables occurring in S range over the set A, and when the symbol 'm', if it occurs in S, is taken to designate the function **m**.

We say that a sentence of L_M is *arithmetically true* if it is satisfied in every model of L_M. And we deal with the arithmetical truth of formulas with free variables by considering the truth of their *closures*. By the closure of a formula we mean the sentence resulting from the formula by adding sufficient universal quantifiers to bind all free variables in the formula. Thus '$(\forall a)(m(a) > 0)$' is the closure of '$m(a) > 0$', and is also the closure of itself.

Using these notions, we may define meaningfulness by means of the following pair of definitions.

Definition 1. An atomic formula S of L_M is empirically meaningful if and only if the closure of the formula

$$\alpha > 0 \rightarrow (S \leftrightarrow S(\alpha))$$

is arithmetically true for every numerical term α, where $S(\alpha)$ results from S by replacing any occurrence of 'm' in S by the term α, followed by the multiplication sign, followed by 'm'.[9] If, for example,

$$S = \text{`}m(a) > m(b)\text{'}$$

$$\alpha = \text{`}(2 + 1)\text{'},$$

[8] Technical details about $\mathfrak{S}$ are omitted. Characterization of models of the purely arithmetical part of L_M are familiar from the literature.

[9] In this definition and subsequently we follow, without explicit discussion, certain use-mention conventions. It would be diversionary to go into these conventions, and it seems unlikely any serious confusion will result from not being completely explicit on this rather minor point.

then

$$S(\alpha) = `(2 + 1)\cdot m(a) > (2 + 1)\cdot m(b)'.$$

Definition 2. *A formula S of L_M is empirically meaningful in sense A if and only if each atomic formula occurring in S is itself empirically meaningful in the sense of Definition* 1.

It is clear on the basis of Definitions 1 and 2 that Sentence 5 is not empirically meaningful in sense A.

On the other hand, there is a certain logical difficulty, within ordinary two-valued logic, besetting the set of true formulas which are meaningful in sense A. Following Tarski [5], a set of formulas is a *deductive system* if and only if the set is closed under the relation of logical consequence, that is, a formula which is a logical consequence of any subset of formulas in the given set must also be in the set. Clearly it is most desirable to have the set of meaningful true formulas about any phenomenon be a deductive system, but we have for the present case the following negative result.

Theorem 1. *The set of formulas of L_M which are meaningful in sense A and whose closures are true is not a deductive system.*

Proof: The true sentence:

$$(\forall x)(x > 2 \rightarrow x > 1)$$

is meaningful in sense A, but the following logical consequence of it is not:

$$m(o_1) > 2 \rightarrow m(o_1) > 1,$$

for the two atomic sentences '$m(o_1) > 2$' and '$m(o_1) > 1$' are both meaningless in the sense of Definition 1.

To be sure, there are some grounds for maintaining that formulas that are empirically meaningless may play an essential deductive role in empirical science, but *prima facie* it is certainly desirable to eliminate them if possible.

A second objection to Definition 1 is that, by considering numerical terms α rather than similarity transformations, we have in effect restricted ourselves to a denumerable number of similarity transformations because the number of such terms in L_M is denumerable. The intuitive idea of invariance with respect to *all* similarity transformations may be caught by a definition of meaningfulness which uses the concept of two models of L_M being related by a similarity transformation. (The operation $\circ$ referred to in the definition is that of functional composition.)

Definition 3. *Let* $\mathfrak{M}_1 = \langle \mathfrak{S}, A_1, \mathbf{m}_1 \rangle$ *and* $\mathfrak{M}_2 = \langle \mathfrak{S}, A_2, \mathbf{m}_2 \rangle$ *be two models of* L_M. *Then* $\mathfrak{M}_1$ *and* $\mathfrak{M}_2$ *are related by a similarity transformation if and only if:*

(i) $A_1 = A_2$.

(ii) *There is a similarity transformation* ϕ *such that*

$$\phi \circ \mathbf{m}_1 = \mathbf{m}_2.$$

Using these notions, we may replace Definitions 1 and 2 by the following:

Definition 4. *A formula S of* L_M *is empirically meaningful in sense B if and only if S is satisfied in a model* $\mathfrak{M}$ *of* L_M *when and only when it is satisfied in every model of* L_M *related to* $\mathfrak{M}$ *by a similarity transformation.*

Unfortunately, we have for meaningfulness in sense B a result analogous to Theorem 1.

Theorem 2. *Let* $\mathfrak{M}$ *be a model of* L_M. *Then the set of all formulas which are meaningful in sense B and which are satisfied in* $\mathfrak{M}$ *is not a deductive system.*

Proof: Consider the two sentences:

(1) $(\forall a)(\forall b)(a = b \rightarrow (m(a) = 2 \rightarrow m(b) = 2))$

(2) $(\forall a)(\forall b)(a = b)$.

It is easy to verify that Sentences 1 and 2 are satisfied in any model whose set A has exactly one element, and are meaningful in sense B, yet they have as a logical consequence the sentence:

(3) $(\forall a)(\forall b)(m(a) = 2 \rightarrow m(b) = 2)$

which is not meaningful in sense B. That this is so may be seen by considering a model with at least two objects with different masses. Let $A = \{o_1, o_2\}$, and let $\mathfrak{M}_1 = \langle \mathfrak{S}, A, \mathbf{m}_1 \rangle$ be such that $\mathbf{m}_1(o_1) = 2$ and $\mathbf{m}_2(o_2) = 3$, and let $\mathfrak{M}_2 = \langle \mathfrak{S}, A, \mathbf{m}_2 \rangle$ be related to $\mathfrak{M}_1$ by the similarity transformation $\phi(x) = 2x$. Thus $\mathbf{m}_2(o_1) = 4$ and $\mathbf{m}_2(o_2) = 6$. It is then easily checked that Sentence 3 is satisfied in $\mathfrak{M}_2$ but not in $\mathfrak{M}_1$.

The negative result of these two theorems indicates the difficulties of eliminating the appearance of empirically meaningless statements in valid arguments with meaningful premises. We return to this point in the next section in connection with consideration of a three-valued logic.

On the other hand, we do have the positive result for both senses of meaningfulness that the set of meaningful formulas is a Boolean algebra; more exactly, the set of such formulas under the appropriate equivalence relation is such an algebra. Here we carry out the construction only for sense B. We consider the theory of Boolean algebras as based on six primitive notions: the nonempty set B of elements; the operation $+$ of addition which corresponds to the sentential connective 'or'; the operation $\cdot$ of multiplication which corresponds to the connective 'and'; the operation $\bar{x}$ of complementation which corresponds to negation; the zero element 0, which corresponds to the set of logically invalid formulas; and the unit element 1, which corresponds to the set of logically valid formulas. We omit stating familiar postulates on these notions which a Boolean algebra must satisfy.

Let E be the set of formulas which are empirically meaningful in sense B. We define the equivalence class of a formula S in E as follows: *$[S]$ is the set of all formulas S' in E which are satisfied in exactly the same models $\mathfrak{M}$ of L_M as S is.* Let $\mathbf{E}$ be the set of all such equivalence classes; obviously $\mathbf{E}$ is a partition of E. The zero element $\mathbf{0}$ is the set of formulas in E which are satisfied in no model of L_M; the unit element $\mathbf{1}$ is the set of formulas in E which are satisfied in all models of L_M. If S and T are in E, then $[S] + [T]$ is the set of all formulas in E which are satisfied in the models of L_M in which either S or T is satisfied. If S and T are in E, then $[S] \cdot [T]$ is the set of all formulas in E which are satisfied in those models in which both S and T are satisfied. Finally if S is in E, then $[\bar{S}]$ is the set of formulas which are satisfied in a model if and only if S is not satisfied in the model. On the basis of these definitions, it is straightforward but tedious to prove the following:

Theorem 3. *The system* $\langle \mathbf{E}, +, \cdot, {}^{-}, \mathbf{0}, \mathbf{1} \rangle$ *is a Boolean algebra.* (The proof is omitted.)

I interpret this theorem as showing that the set of meaningful formulas in sense B of L_M has a logical structure identical with that of classical logic. In connection with other systems of measurement for which the set of transformations referred to in the analogue of Definition 3 is not a group, this classical Boolean structure does not necessarily result.

Exponentiation was introduced into L_M deliberately to illustrate the sensitivity of the decidability of meaningfulness to the strength of L_M. The problem of decidability for the arithmetical language of Tarski's monograph mentioned earlier is open when his language is augmented by notation for exponentiation to a fixed base. It seems unlikely that the decidability of meaningfulness in L_M can be solved without solving this more general problem. If L_M is weakened by deleting exponentiation to the base 2, then it easily follows from Tarski's well-known result that meaningfulness is decidable. On the other hand, if L_M is strengthened to include sufficient elementary number theory to yield undecidability of whether, for instance, a given term designates zero, then meaningfulness is not decidable, for the meaningfulness of formulas of the form $m(a) = \mathfrak{t}$, where $\mathfrak{t}$ is a numerical term, would not be decidable.

A THREE-VALUED LOGIC FOR L_M

Since sentences like '$(\forall a)(m(a) > 2)$' of L_M cannot be determined as true or false on the basis either of logical argument or of empirical observation, it is natural to ask what are the consequences of assigning them the truth value *meaningless*, which we designate by 'μ', and reserving the values *truth* and *falsity* for meaningful sentences, which we designate by 'T' and 'F' respectively. The first thing to be noticed is that meaningfulness in sense B does not lead to a truth-functional logic in these three values. This may be seen by considering two examples. The component sentences of the sentence:

$$(\exists a)(m(a) = 1) \vee -(\exists a)(m(a) = 1)$$

have the value μ but the whole sentence is meaningful in sense B and has the value T. On the other hand, the component sentences of:

$$(\exists a)(m(a) = 1) \vee (\exists b)(m(b) = 2)$$

have the value μ and so does the whole sentence. Thus these two examples taken together show that disjunction is not truth-functional for a three-value logic of meaningfulness in sense B.

The state of affairs for meaningfulness in sense A is much better: it does lead to a truth-functional logic in the three values T, F, and μ. The appropriate truth tables are easily found by using the simple observation that a formula has the value μ if any well-formed part of it has that value. Thus as the tables for negation and conjunction we have:

S	$-S$
T	F
F	T
μ	μ

&	T	F	μ
T	T	F	μ
F	F	F	μ
μ	μ	μ	μ

Tables for the sentential connectives of disjunction, implication, and equivalence follow at once from the standard definitions of these connectives in terms of negation and conjunction. On the other hand, it is obvious that this three-valued logic is not functionally complete with respect to negation and conjunction. For example, we cannot define in terms of these two connectives a unary connective which assigns the value μ to formulas having the value T.

Besetting meaningfulness in sense A is the negative result of Theorem 1. This difficulty we shall meet head on by proposing a revision of the definition of the semantical notion of logical consequence. However, before turning to this definition, it will be advantageous to give a model-theoretic definition of meaningfulness which combines the virtues of sense A and sense B.

Definition 5. *A formula S of L_M is empirically meaningful in sense C if and only if every atomic formula occurring in S is meaningful in sense B.*

It is easily verified that the truth tables just given are satisfied when the value μ signifies meaninglessness in sense C. Moreover, the exact analogue of the Boolean structure theorem for sense B (Theorem 3) can be proved for sense C.

To meet the difficulty of having formulas which are meaningless in sense C be logical consequences of formulas which are meaningful in sense C, a revision of the standard definition of *logical consequence* is proposed. For this purpose we need to widen the notion of a model to that of a *possible realization* of L_M. A model of L_M requires that the arithmetical symbols be interpreted in terms of the usual system of real numbers, but no such restriction is imposed on a possible realization. For example, any domain of individuals and any two binary operations on this domain provide a possible realization of the operation symbols of addition and multiplication. Details of the exact definition of a possible realization are familiar from the literature and will not be given here. This notion is used to define that of logical consequence, namely, a formula S of L_M is a *logical consequence* of a set A of formulas of L_M if S is satisfied in every possible realization in which all formulas in A are satisfied. We may then define:

Definition 6. *Let S be a formula and A a set of formulas of L_M. Then S is a meaningful logical consequence of A if and only if S is a logical consequence of A and S is meaningful in sense C whenever every formula in A is meaningful in sense C.*

The central problem in connection with this definition is to give rules of inference for which it may be established that, if A is a set of formulas meaningful in sense C, then S is a meaningful logical consequence of A

if and only if S is derivable from A by use of the rules of inference.[10] For this purpose, we may consider any one of several systems of natural deduction. The eight essential rules are: rule for introducing premises; rule for tautological implications; rule of conditional proof (the deduction theorem); rule of universal specification (or instantiation); rule of universal generalization; rule of existential specification; rule of existential generalization; and rule governing identities.[11] To these eight rules we add the *general restriction* that every line of a derivation must be a formula meaningful in sense C. This means, for instance, that in deriving a formula by universal specification from another formula we must check that the result of the specification is meaningful. This restriction entails that the modified rules of inference are finitary in character only if there is a decision procedure for meaningfulness in sense C. Remarks on this problem were made at the end of the previous section. Because we have modified the rules of inference only by restricting them to meaningful formulas, it follows easily from results in the literature on the soundness of standard rules of inference that:

Theorem 4. Let A be a set of formulas meaningful in sense C. If a formula S is derivable from A by use of the rules of inference subject to the general restriction just stated, then S is a meaningful logical consequence of A.

Of considerable more difficulty is the converse question of completeness, namely, does being a meaningful logical consequence of a set of meaningful formulas imply derivability by the restricted rules? The following considerations suggest that the answer may be affirmative to this question. Let $L_M{}^*$ be a second language which differs from L_M in the following single respect: the one-place function symbol 'm' is replaced by the two-place function symbol 'r', where both argument places are filled by individual variables or constants. The intuitive interpretation of the formula '$r(a, b) = x$' is that the numerical ratio of the mass of a to the mass of b is the real number x, that is,

$$r(a, b) = m(a)/m(b).$$

Clearly every formula in $L_M{}^*$ is meaningful with respect to our intuitive criterion of invariance. (The practical objection to $L_M{}^*$ is that

[10] Although two kinds of variables are used in L_M, we may easily modify L_M to become a theory with standard formalization in first-order predicate logic and thus consider only modification of standard rules of inference for first-order predicate logic.

[11] By various devices this list can be reduced, but that is not important for our present purposes. Exposition of systems of natural deduction which essentially use these eight rules is to be found in [6], [7], and [8].

such a ratio language is tedious to work with and does not conform to ordinary practice in theoretical physics.) No restrictions on the rules of inference are required for L_M^* and, consequently, the usual completeness result holds. The suggestion is to use translatability of meaningful formulas of L_M into L_M^* to prove completeness of inferences from meaningful formulas of L_M. The possible pitfall of this line of reasoning is that translatability requires certain arithmetical operations which are preserved in every model but not necessarily in every possible realization of L_M.

Certain aspects of this construction of a three-valued logic for L_M seem worthy of remark. In the first place, the construction has assumed throughout use of a two-valued logic in the informal metalanguage of L_M. In particular, ordinary two-valued logic is used in deciding if a given sentence of L_M is satisfied in a given model of L_M. On the other hand, the relation between sets of empirical data on mass measurements and models of L_M is one–many. The empirical content of the data is expressed not by a particular model but by an appropriate equivalence class of models. Consequently, sentences of L_M which are not invariant in truth value (in the two-valued sense) over these equivalence classes do not have any clear empirical meaning even though they have a perfectly definite meaning relative to any one model. Thus it seems to me that to call a formula like '$m(a) = 5$' empirically meaningless is no abuse of ordinary ideas of meaningfulness, and in this particular situation accords well with our physical intuitions. If this is granted, the important conclusion to be drawn is that, for the language L_M, the three-valued logic constructed is intuitively more natural than the ordinary two-valued one.

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