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DISCUSSION

NELSON GOODMAN ON THE CONCEPT OF LOGICAL SIMPLICITY

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Nelson Goodman is undoubtedly solely responsible for raising the concept of logical simplicity to a status where it can be taken seriously by logicians and methodologists. I find myself in complete agreement with the general direction of his work; however, on many questions of detail I differ with the analysis given in his recent paper "Axiomatic measurement of simplicity."[1]

My comments on Goodman's paper naturally fall into three categories: the problem of dependent primitives, relevant kinds and emphasis on single predicate bases, and methodological status of his axioms. Generally speaking, I am in considerably more agreement with the paper by John Kemeny which accompanies Goodman's,[2] and I shall refer to Kemeny's paper for comparative purposes.[3]

In my discussion I am assuming that the basic intuitive idea to be captured formally is that of *structural simplicity*. An entirely different notion of simplicity could concern itself with the relative complexity of predicate bases for the same structure, for instance, an evaluation of different sets of primitive notions for groups or Boolean algebras. Kemeny's two measures assign the same complexity value to different predicate bases yielding the same structure. My own feeling is that Goodman's paper suffers from a failure to distinguish clearly these two different problems. My comments are concerned with structural simplicity rather than relative simplicity of different axiomatizations of the same structure.

1. The Problem of Dependent Primitives. In the several discussions of multi-predicate bases in Goodman's writings, the problem of bases with dependent predicates is not examined. Yet some very counter-intuitive results may be constructed by considering dependent predicates.[4]

[1] *J. of Philosophy*, vol. 52 (1955), pp. 709–722.

[2] *J. of Philosophy*, vol. 52 (1955), pp. 722–733.

[3] The content of my comments constitute the body of my remarks on Goodman's and Kemeny's papers at the Symposium on the Concept of Logical Simplicity of the American Philosophical Association at Boston University on December 28, 1955.

[4] The predicates of a basis are independent if none is definable in terms of the other. The standard method of proving predicates independent is by use of Padoa's principle. Roughly speaking, a predicate of a basis is independent of the other predicates if two models satisfying the postulates of the basis can be found such that in both models each predicate except the given one has an identical interpretation, and the given predicate has different interpretations. It is remarkable that nowhere in Goodman's writing does there seem to be any mention of Padoa's principle. For further discussion and justification of the principle, see J. C. C. McKinsey, "On the independence of undefined ideas," *Bull. of the Amer. Math. Soc.*, vol. 41 (1935), pp. 291–297, and A. Tarski, "Einige methodologische Untersuchungen über die Definierbarkeit der Begriffe," *Erkenntnis*, vol. 5 (1935–1936), pp. 80–100.

Let us plunge into an example and tidy up afterward. Consider the basis B_1, consisting of the single binary predicate 'Q_1', which is reflexive (totally reflexive in Goodman's terminology) and transitive. (In mathematical language, 'Q_1' designates a quasi ordering.) Second, consider the basis B_2 consisting of two binary predicates, 'Q_2' and 'C_2'. The predicate 'Q_2', like 'C_2', designates a quasi-ordering; moreover the two predicates 'Q_2' and 'C_2' are connected by the postulate:

$$(x)(y)(xC_2y \leftrightarrow xQ_2y \;\&\; yQ_2x).$$

It follows from Goodman's Postulate P2, which stipulates that every predicate has a positive complexity value and that the value of a basis is the sum of the values of the predicates in it, that the complexity value of B_2 is greater than the complexity value of B_1. On the other hand, it is obvious that B_1 and B_2 are mutually replaceable. Intuitively, bases B_1 and B_2 express an equal amount, since we may define a predicate 'C_1' related to 'Q_1' in the way that 'C_2' is related to 'Q_2'.

My objection to valuing B_2 higher than B_1 is that dependent predicates do not add anything new and thus should not contribute to the complexity value. It is a definite merit of Kemeny's two measures of complexity that they both assign the same value to B_1 and B_2, and to other pairs of bases similarly related regarding dependencies.

Moreover, a very general result of Tarski's shows that a wide class of bases may be replaced by and replace a basis consisting of a single binary predicate. Roughly speaking, his result is that if a universal pair predicate is definable in terms of the primitive predicates then such a single binary predicate can be found.[5]

By means of Tarski's theorem we thus obtain the unsatisfactory result that we can find two mutually replaceable bases whose complexity values differ by more than any fixed amount no matter how great. (Note that I am here referring to replaceability as a relation between bases independent of any reference to Goodman's relevant kinds.)

2. Relevant Kinds and Emphasis on Single Predicate Bases. It seems to me that in stipulating "that interconnections among the predicates in a basis are not factors admitted in the definition of relevant kinds" Goodman has nearly eliminated the usefulness of his construction of complexity values. (Roughly speaking, a relevant kind is a class of predicate bases having the same general structure. For example, the class of bases consisting of a single symmetric predicate is a relevant kind.) Only the most trivial or awkward axiomatized theories are based on a single predicate. And among those based on several predicates only the most trivial of these do not postulate connections between their basic predicates. Let me be clear about this point. As Tarski's

[5] See A. Tarski, "A general theorem concerning the reduction of primitive notions," (Abstract), *Journal of Symbolic Logic*, vol. 19 (1954), pp. 158–159. I am indebted to Leon Henkin for this reference. The precise definition of a universal pair predicate 'C' is:

$$(x)(y)(\exists z)(u)(uCz \leftrightarrow u = x \vee u = y).$$

result shows, for a very wide class of theories a single binary predicate suffices. But in practice no one is interested in basing a theory on these necessarily odd predicates. Assuming that questions of simplicity are interesting (as I think they are), what is wanted is a measure which applies in a substantial way to theories in their standard axiomatic forms. It is, I think, significant that in all of Goodman's writings on simplicity substantive examples are scarcely discussed. Predicate bases for the axiomatic theory of groups, semi-groups, groupoids, rings, fields, Boolean algebras and the like are not analyzed and evaluated.

If such substantive examples, easily axiomatizable within the first order predicate calculus with identity, are not discussed, it would seem necessary at least to consider the effect on complexity values of such recurring properties as arise, say, from one relation being a congruence relation for another, an operation being monotonic with respect to a relation, or an operation having the cancellation property with respect to a relation. (A phrasing of properties like these three in terms of predicates rather than operations and relations is easily given.) Yet what we find is that Goodman's crucial postulates P3 and P4 deal only with bases consisting of a single predicate.[6] Certainly n-place predicates are of some interest to logicians, but it is hard to think of an interesting mathematical example with $n \geqq 5$, and even 4-place predicates are none too common.

Again, it is a merit of Kemeny's two measures of complexity that they are not subject to this defect of emphasizing single predicates. A simple example will illustrate the difference. Let basis B_3 consist of a transitive predicate 'Q_3' and a reflexive, symmetric and transitive predicate 'C_3'. Let basis B_4 consist of a transitive predicate 'Q_4', a reflexive, symmetric and transitive predicate 'C_4' and let 'Q_4' and 'C_4' satisfy the postulate:

$$(x)(y)(u)(v)(xC_4u \,\&\, yC_4v \,\&\, xQ_4y \rightarrow uQ_4v).$$

(Thus, intuitively, C_4 is a congruence relation for Q_4, whereas C_3 is merely an arbitrary equivalence relation.) Then on Goodman's analysis B_3 and B_4 have exactly the same complexity value and belong to the same relevant kind. On the other hand, Kemeny's two measures both yield the intuitively satisfactory result that the complexity of B_3 is greater than that of B_4.

3. Methodological Status of Goodman's Axioms. As the title of Goodman's article indicates, he is concerned to give an *axiomatic* account of simplicity. From the standpoint of the general methodology of axiomatics there are, I believe, a number of defects in his system. I do not have in mind standards requiring a formalized language; such a requirement would certainly be too stringent in this context. Rather I have in mind ordinary mathematical standards for axiomatizing a theory—the kind of standards one would apply in

[6] Even the emphasis within single predicate bases seems distorted. Certainly one of the most important properties for classifying ternary predicates is that of designating a *binary operation*, which is not mentioned by Goodman.

Goodman's Postulate P3 is quoted below, and P4 is an extension of P3.

analyzing axioms for group theory or topology, say, or just as well, mechanics or learning theory. In addition, I want to make some remarks specific to axiomatic theories of measurement.[7]

Axiomatization in the ordinary mathematical sense naturally divides into four parts. First, a statement of what other theories are assumed is needed. There is no difficulty about Goodman's work here; since his complexity function is real-valued it is clear he wants to assume ordinary logic and a good part of standard mathematics.

Second, the primitive notions of the theory on which the axioms are based should be listed and their set-theoretical (or general mathematical) structure stated. For example, in axiomatizing the theory of groups we need two primitive notions: a set G and a binary operation $*$ from the Cartesian product $G \times G$ to G.[8] It would appear (with the reservation stated below) that Goodman intends his axioms to be based on a single primitive, namely, his complexity measure v. However, the situation is ambiguous. All notions used in axioms and theorems of a theory must be either (i) primitive notions, (ii) notions occurring in the theories assumed (here logic and mathematics), or (iii) notions explicitly definable in terms of notions falling under (i) or (ii). The notion whose status is ambiguous in Goodman's axiomatization is that of relevant kind. This is not a standard notion of logic, nor is it explicitly defined in the paper. Various remarks are made regarding the intuitive idea but no exact characterization is given. From this lacuna, one might conclude that Goodman meant this notion to be a primitive notion, and not a notion of logic defined for the purposes of his article. However, this alternative entails serious difficulties, for predicate bases and relevant kinds are related by the axioms only in the most meagre way: relevant kinds are sets of predicate bases. This bare fact is inadequate to prove most of the theorems listed. The theorems can in fact only be proved by ascertaining in a catch-as-catch-can fashion what properties relevant kinds are supposed to possess. From Goodman's intuitive hints it seems clear that a precise and adequate definition of relevant kinds is going to be tedious and awkward. Kemeny makes no use of the notion in defining either one of his complexity measures. My own opinion is that it is a notion best dispensed with.

The third part of an axiomatization is to state the axioms (or postulates) of the theory. In this connection I want to remark on the somewhat peculiar formulation of Goodman's third and fourth postulates. Since similar remarks apply to both of them, it will suffice to consider only P3, which reads:

> *If two relevant kinds K and L are like those specified in* T*15 except that* $(vK - vL)/c$ *is not numerically determined by* P1 *and* P2, *then* $vK - vL = syL - syK$.

[7] An elaboration of the general viewpoint I am adopting here is to be found in my article "Some remarks on problems and methods in the philosophy of science," *Philosophy of Science*, vol. 21 (1954), pp. 242–248.

[8] If a formalized language were being used, such as is envisaged for the predicate bases whose simplicity Goodman studies, then it would be natural to speak of a one-place predicate 'G' and a two-place functor '$*$'.

We look at "Theorem" T15 (Goodman says he has not yet found a proof of it), and find it reads:

> *If K and L are relevant kinds of n-place irreflexive predicates and* $scK = scL$, *and* $sy\,K \leqq sy\,L$, *and* $(vK - vL)/c$ *is numerically determined by* P1 *and* P2, *then* $vK - vL = sy\,L - sy\,K$.

Finally, preceding T15, we find "numerically determined" defined: "determined as representable by a specific purely numerical expression, free of signs for unknown quantities." In view of the absence of a proof of T15 it would seem much preferable to eliminate references to the vaguely defined notion of numerically determined, and simply formulate P3 as:

> P3′. *If K and L are relevant kinds of n-place irreflexive predicates, and* $sc\,K = sc\,L$ *and* $sy\,K \leqq sy\,L$, *then* $vK - vL = sy\,L - sy \leqslant K$.

In standard axiomatics it is not customary for one postulate explicitly to refer to another, nor for a variety of reasons is it desirable to introduce a metamathematical notion such as *numerically determined* when the language of the axiomatization is not formalized. The reformulation P3′ suffers from neither of these defects.

A similar reformulation of P4 is easily provided and so will be omitted here.

The fourth part of an axiomatization of a theory concerns the proofs of theorems. What is most desirable are structure theorems characterizing as closely as possible the theory which has been axiomatized. In this connection I would like to turn from general methodological comments to specific remarks concerning axiomatizations of theories of measurement. My main point is that Goodman's fifth and last postulate is mistaken in conception and irrelevant to his theory. Postulate P5 reads:

> *Just those kinds that are determined by preceding postulates to have the same value as* $\{1 - pl\}/$ *have the value* 1*; and every other kind has the lowest integral value consistent with this requirement and preceding postulates.*

Both parts of this postulate seem to be motivated by the mistaken conception that postulates for measurement of an attribute must yield a unique assignment of numbers. The opposite is in fact the case, for it is important and significant to distinguish between formally different scales of measurement. For example, a temperature measurement by means of an ordinary thermometer is only significant up to a positive linear transformation, that is, the same physical facts about temperature will be given by any temperature scale which is obtained from the original one by multiplying by a positive number and adding an arbitrary number (the Fahrenheit and Centigrade scales are so related). By comparison, a measurement of mass is "tighter", in the sense that it is significant up to a similarity transformation, that is, multiplication by a positive number (weight measurements in terms of grams and pounds are so related). Now the first part of Goodman's P5 corresponds to arbitrarily selecting a unit, and my contention is that in introducing this choice of a unit into his postulates he has confused two different things, namely, the axiomatic problem of formulating the intuitively essential conditions an adequate complexity measure should satisfy and the different problem of choosing one among the possibly large number of measures satisfying the essential conditions. This latter choice is merely a matter of convenience and yields nothing interesting regarding the notion of simplicity.

The same remarks apply to the second half of P5. Among the presumably infinite number of measures satisfying P1–P4 and the first half of P5, he selects one yielding integral values. For working purposes this selection is convenient, but we should not be misled into thinking that this choice expresses anything significant about simplicity, as the preceding postulates do.[9]

Dispensing with P5, two basic theorems are called for: one theorem establishing that there is a complexity measure satisfying P1–P4, which in this case amounts to showing that the axioms are consistent, and a second theorem asserting how unique is a complexity measure satisfying P1–P4.[10] To be explicit, this second theorem would read:

> *Let v_1 and v_2 be two complexity measures satisfying* P1–P4. *Then v_1 and v_2 are related by a blank transformation.*

My conjecture is that finding the correct group of transformations is difficult. Pretty clearly it is wider than the group of similarity transformations, so 'blank' cannot be replaced by 'similarity'. Until this group is determined, however, it is by no means evident in any formal, proper sense what general type of measurement of simplicity Goodman has proposed.

Finally, to show that the first kind of theorem is not always trivial to establish, I would like to conclude my comments by formulating a weak set of axioms for simplicity, for which the existence of an appropriate complexity measure is an open problem of considerable difficulty.

Let P be a non-empty finite set of extralogical predicates and let A be a non-empty, finite, consistent set of axioms using only extralogical predicates which are members of P. A *predicate basis* is then an ordered pair (P_1, A_1) where P_1 is a subset of P and A_1 is a subset of A such that the only extralogical predicates used in axioms which are members of A_1 are members of P_1.[11] The only primitive notion of the axiomatization is a binary relation $\preccurlyeq$, whose field is the set of all bases constructed from P and A. The intended interpreta-

[9] It should be clear that these criticisms of Goodman's P5 do not apply to Kemeny's two measures, for Kemeny does not attempt an axiomatic analysis, but only proposes two particular measures. As a matter of fact, the desired properties of Kemeny's two measures would be preserved by any similarity transformation.

[10] Statements and proofs of such a pair of theorems for axiomatizations of different branches of the theory of measurement are to be found in my article, "A set of independent axioms for extensive quantities," *Portugaliae Mathematica*, vol. 10 (1951), pp. 163–172, and my joint article with Muriel Winet, "An axiomatization of utility based on the notion of utility differences," *Management Science*, vol. 1 (1955), pp. 259–270.

[11] I am a little uneasy with both Goodman's and Kemeny's definitions of predicate bases. The formulation just stated attempts to be a little more precise.

tion is that $(P_1, A_1) \preccurlyeq (P_2, A_2)$ if and only if the basis (P_1, A_1) is *not more complex* than (P_2, A_2). We now state:

A relation $\preccurlyeq$ is a *complexity relation* for the bases constructed from P and A if and only if the following seven axioms are satisfied for every basis (P_1, A_1), (P_2, A_2), (P_3, A_3):

S1. *If* $(P_1, A_1) \preccurlyeq (P_2, A_2)$ *and* $(P_2, A_2) \preccurlyeq (P_3, A_3)$ *then* $(P_1, A_1) \preccurlyeq (P_3, A_3)$.
S2. *Either* $(P_1, A_1) \preccurlyeq (P_2, A_2)$ *or* $(P_2, A_2) \preccurlyeq (P_1, A_1)$.
S3. *If* $A_1 = A_2$ *and* P_1 *is a subset of* P_2 *then* $(P_1, A_1) \preccurlyeq (P_2, A_2)$.
S4. *If* $P_1 = P_2$ *and* A_1 *is a subset of* A_2 *then* $(P_2, A_2) \preccurlyeq (P_1, A_1)$.
S5. *If* $P_1 \cap P_3 = P_2 \cap P_3 = \Lambda$ *and* $(P_1, A_1) \preccurlyeq (P_2, A_2)$ *then* $(P_1 \cup P_3, A_1 \cup A_3) \preccurlyeq (P_2 \cup P_3, A_2 \cup A_3)$.
S6. $(\Lambda, \Lambda) \preccurlyeq (P_1, A_1)$.
S7. *It is not the case that* $(P, A) \preccurlyeq (\Lambda, \Lambda)$.[12]

The intuitive content of each of the axioms should be clear. It is easy to show that the comparative relation generated by either of Kemeny's two measures satisfies the axioms. But the open problem is: given any complexity relation $\preccurlyeq$ satisfying the axioms, does there exist a numerical complexity measure f such that:

(i) $f(P_1, A_1) \leqq f(P_2, A_2)$ *if and only if* $(P_1, A_1) \preccurlyeq (P_2, A_2)$,
(ii) *If* $P_1 \cap P_2 = \Lambda$ *then*

$$f(P_1 \cup P_2, A_1 \cup A_2) = f(P_1, A_1) + f(P_2, A_2)?$$[13]

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[12] The simple notation used in the axioms should be familiar. Thus, Λ is the empty set; $P_1 \cap P_2$ is the intersection of the sets P_1 and P_2, that is, the set of all predicates common to P_1 and P_2; $P_1 \cup P_2$ is the union of P_1 and P_2, that is, the set of all predicates which are either in P_1 or in P_2.

[13] The openness of this problem follows from the openness of a corresponding problem in the theory of subjective probability. The problem was first posed by de Finetti. For a discussion, see L. J. Savage, *Foundations of Statistics*, New York, 1954, Chapter 3.