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NEW BELL-TYPE INEQUALITIES FOR $N > 4$ NECESSARY FOR EXISTENCE OF A HIDDEN VARIABLE

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The purpose of this paper is to extend Bell's inequalities to obtain some general necessary conditions for the existence of a joint probability distribution for any finite collection of Bell-type random variables. Our results show that for $N > 4$ many new elementary inequalities beyond those of Bell must be satisfied by any hidden variable theory.

Key words: Bell-type inequalities, hidden variables.

1. INTRODUCTION

Fine [1] proved that Bell's [2] inequalities are sufficient, as well as necessary, to guarantee the existence of a joint probability distribution of the random variables in question, e.g., those representing certain spin correlation experiments. Garg and Mermin [3] gave an example of eight random variables satisfying Bell's inequalities but such that no joint distribution exists.

The purpose of this paper is to extend Bell's inequalities to obtain some general necessary conditions for the existence of a joint probability distribution for any finite collection of Bell-type random variables. Our results show that, for $N > 4$, many new elementary inequalities beyond those of Bell must be satisfied by any hidden-variable theory. Since these additional inequalities are violated by quantum mechanical predictions for appropriate choice of measurement arrangements, they serve to increase the conceptual distance

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between what may be called Einstein locality, after the EPR paradox, and quantum mechanics.

To give a concrete sense of the nature of our results, we exhibit one of the new inequalities violated by Garg and Mermin's example. For theoretical purposes it will suffice to introduce Bell-type random variables $X_1, \ldots, X_N$, $N \geq 4$, having possible values ± 1, with means $E(X_i) = 0$ and with what we term *Bell covariances* $E(X_i X_j)$ relative to the index i_0, with $1 \leq i \leq i_0 < j \leq N$, $2 \leq i_0 \leq N-2$. Notice that, when $N = 4$, we must, as is familiar, have $i_0 = 2$. Garg and Mermin's counterexample is for $N = 8$ and $i_0 = 4$. Let $E(X_1 X_5) = E(X_2 X_6) = E(X_3 X_7) = 1$ and otherwise $E(X_i X_j) = -1/3$ for $1 \leq i \leq i_0 < j \leq N$. Then it is easy to show that, for the quintuple $(X_1, X_3, X_4, X_6, X_8)$, all covariances must be $-1/3$. No joint distribution can exist for such covariances—a fact that can be computed directly but also follows from Theorem 1 proved below.

As an example, we give a general inequality for $N = 8$ and $i_0 = 4$ which Garg and Mermin's example violates. We use the same five variables they did, but, as will be seen explicitly later, our new Bell-type inequalities use the full set of $N = 8$ variables. Here is the example inequality, where, to simplify notation, we write B_{ij} for $E(X_i X_j)$:

$$-2B_{15} + B_{16} + B_{18} - B_{26} + B_{28} + B_{35} + B_{36} - B_{37} + B_{38} + B_{45} + B_{46} + B_{47} + B_{48} \geq -6.$$

It is easy to check that, for the Garg and Mermin example, the right-hand side of the inequality is $-7\ 1/3$, and so it is violated.

We do emphasize that, even for $N = 8$, our full necessary set of new Bell-type inequalities is large, but, as we shall see, the schema representing them is quite simple. In Sec. 2 we prove a general theorem on the existence of a joint probability distribution when *all* covariances are given, not just the Bell ones. This theorem is needed in Sec. 3 when we turn to the main theorem of the paper on Bell-type inequalities for $N > 4$.

Finally, we remark that we conjecture our necessary condition is also sufficient for the existence of a joint probability distribution for $N > 4$, but Pitowsky [4] has given a general complexity argument that is a basis for being skeptical of this conjecture.

2. A PRELIMINARY THEOREM

We prove by a simple argument an inequality that follows from the existence of a joint distribution of N random variables. The proof that the inequality for the subsets of odd number is sufficient is much more complicated and is not needed here.

Theorem 1. *Let* $X_1, \ldots, X_N$ *be* N *random variables with* $X_i = \pm 1$, $E(X_i) = 0$ *and given covariances* $E(X_i X_j)$, $1 \leq i < j \leq N$. *Then , if a joint distribution of* $X_1, \ldots, X_N$ *exists, we have*

$$\sum_{i<j} X_i X_j E(X_i X_j) \geq \begin{cases} \dfrac{1-|J|}{2}, & |J| \text{ odd}, \\ \dfrac{-|J|}{2}, & |J| \text{ even}, \end{cases} \tag{1}$$

for i, j *in* J, *a nonempty subset of* $\{1, \ldots, N\}$ *and* $|J|$ *the cardinality of* J.

Proof. We prove (1) holds for every realization $(x_1, \ldots, x_N)$ of the random variables, where each $x_i = \pm 1$. For $|J|$ odd, since $\sum_{i \in J} x_i X_i$ cannot be zero, we have

$$1 \leq E\Big(\sum_{i \in J} x_i X_i\Big)^2 = E\Big(|J| + 2\sum_{i<j} x_i x_j X_i X_j\Big)$$

$$= |J| + 2\sum_{i<j} x_i x_j E(X_i X_j),$$

and, in case $|J|$ is even, we have

$$0 \leq E\Big(\sum_{i \in J} x_i X_i\Big)^2.$$

In both cases, (1) is then immediate.

The theorem is easily generalized to random variables which are restricted only to having expectations that are zero and variances that are one, but this generalization is not needed here.

The following corollary whose partial proof from (1) is obvious, will be useful later.

Corollary. *For* $N = 3$, *the inequality*

$$X_1 X_2 E(X_1 X_2) + X_1 X_3 E(X_1 X_3) + X_2 X_3 E(X_2 X_3) \geq -1 \tag{2}$$

is necessary and sufficient for existence of a joint probability distribution of X_1, X_2, and X_3 compatible with the given covariances.

Proof. The necessity of (2) follows at once from Theorem 1. Sufficiency can be proved as follows. Suppes and Zanotti [5] proved that the following inequality is necessary and sufficient:

$$\begin{aligned} -1 &\leq E(X_1X_2) + E(X_1X_3) + E(X_2X_3) \\ &\leq 1 + 2\min\{E(X_1X_2), E(X_1X_3), E(X_2X_3)\}, \end{aligned} \tag{3}$$

and it is easy to show that (2) implies (3).

3. NEW BELL-TYPE INEQUALITIES

We define a *Bell triple* to be three random variables such that exactly the first two have indices either less or greater than i_0, i.e., $1 \leq i < j \leq i_0 < k \leq N$ or $1 \leq k \leq i_0 < i < j \leq N$. For each Bell triple (X_i, X_j, X_k), we define Bell fur :tions B_{ij}:

$$B_{ij}(x_ix_j) = \begin{cases} 1 - \max_k |E(X_iX_k) - x_ix_jE(X_jX_k)|, \\ \text{if } (X_i, X_j, X_k) \text{ is a Bell triple,} \\ x_ix_jE(X_iX_j), \qquad \text{if } 1 \leq i \leq i_0 < j \leq N, \end{cases}$$

where $x_i, x_j = \pm 1$. Intuitively the Bell functions provide upper bounds on the non-Bell covariances, but, of course, in the case of the Bell covariances, the Bell functions for given indices are just identical to them, modulo the sign of x_ix_j, as can be seen from the second branch of the defining equation. The rationale of the Bell functions becomes clear in the proof of the next theorem, the main one in this paper.

Theorem 2. *Let $E(X_iX_j)$ be given Bell covariances relative to the index i_0, $1 \leq i \leq i_0 < j \leq N$, for the random variables $X_1, \ldots X_N$. If there exists a joint distribution of $X_1, \ldots, X_N$ compatible with the given Bell covariances, then, for every nonempty subset J of $\{1, \ldots, N\}$, we have*

$$\sum_{i<j} B_{ij}(x_ix_j) \geq \begin{cases} \dfrac{1-|J|}{2}, & \text{if } |J| \text{ odd,} \\ \dfrac{-|J|}{2}, & \text{if } |J| \text{ even.} \end{cases} \tag{4}$$

for $i, j \in J,\ x_i, x_j = \pm 1$.

Proof. If a joint distribution for the random variables $X_1, \ldots, X_N$ compatible with the given Bell covariances relative to index i_0 exists, then there exists a joint distribution for each Bell triple (X_i, X_j, X_k) with $E(X_iX_j)$ independent of the choice of X_k. Following the Corollary of Theorem 1, we then have, for any realization (x_i, x_j, x_k),

$$x_ix_jE(X_iX_j) + x_ix_kE(X_iX_k) + x_jx_kE(X_jX_k) \geq -1.$$

So, in particular, if $x_i = x_j = 1$, then

$$E(X_iX_j) + x_kE(X_iX_k) + x_kE(X_jX_k) \geq -1; \tag{5}$$

and, if $x_i = 1$, $x_j = -1$, then

$$-E(X_iX_j) + x_kE(X_iX_k) - x_kE(X_jX_k) \geq -1. \tag{6}$$

From (5) and (6) we infer

$$\begin{aligned} -1 - x_k(E(X_iX_k) + E(X_jX_k)) &\leq E(X_iX_j) \\ &\leq 1 + x_k(E(X_iX_k) - E(X_jX_k)), \end{aligned}$$

for every Bell triple (X_i, X_j, X_k) and every $x_k = \pm 1$. Thus we have at once

$$\begin{aligned} -1 + |E(X_iX_k) + E(X_jX_k)| &\leq E(X_iX_j) \\ &\leq 1 - |E(X_iX_k) - E(X_jX_k)|, \end{aligned} \tag{7}$$

and so

$$\begin{aligned} -1 + \max_k |E(X_iX_k) + E(X_jX_k)| &\leq E(X_iX_j) \\ &\leq 1 - \max_k |E(X_iX_k) - E(X_jX_k)|. \end{aligned}$$

If $x_ix_j = 1$, we have

$$1 - \max_k |E(X_iX_k) - E(X_jX_k)| \geq x_ix_jE(X_iX_j); \tag{8}$$

and, if $x_ix_j = -1$,

$$1 - \max_k |E(X_iX_k) + E(X_jX_k)| \geq x_ix_jE(X_iX_j). \tag{9}$$

Combining (8) and (9), we have for the Bell functions

$$B_{ij}(x_ix_j) = 1 - \max_k |E(X_iX_k) - x_ix_jE(X_jX_k)| \geq x_ix_jE(X_iX_j). \tag{10}$$

The desired inequalities (4) follow at once from (10) and (1) of Theorem 1.

Corollary. *If there exists a hidden variable λ such that $X_1, \ldots,$ X_N are conditionally independent, given λ, i.e.,*

$$E(X_1 \cdots X_N \mid \lambda) = E(X_1 \mid \lambda) \cdots E(X_N \mid \lambda),$$

then inequalities (4) *must be satisfied.*

Proof. Suppes and Zanotti [5] proved that the assumption of a hidden variable as just formulated is equivalent to the assumption of a joint probability distribution of $X_1, \ldots, X_N$. The Corollary follows at once from the theorem and this equivalence.

To show how inequalities (4) work in a particular case, we derive the inequality stated at the beginning which rules out Garg and Mermin's counter-example. Remember $N = 8$, $i_0 = 4$, $J = \{1,3,4,6,8\}$, $B_{15} = B_{26} = B_{37} = 1$, and $B_{16} = B_{18} = B_{36} = B_{38} = B_{46} = B_{48} = -1/3$.

We need to compute the Bell functions. As a simple case, we take $x_i = x_j = 1$. So we need only to compute B_{13}, B_{14}, B_{34}, and B_{68}. We give details only for B_{13}:

$$B_{13} = 1 - \max_k |E(X_1X_k) - E(X_3X_k)| = 1 - E(X_1X_5) + E(X_3X_5) = -1/3,$$

and, similarly,

$$B_{14} = 1 - E(X_1X_5) + E(X_4X_5) = -1/3,$$

$$B_{34} = 1 - E(X_3X_7) + E(X_4X_7) = -1/3,$$

$$B_{68} = 1 - E(X_2X_6) + E(X_2X_8) = -1/3.$$

The full inequality in terms of Bell covariances is then as given in the beginning of the paper.

We can also easily derive the Bell inequality for $N = 4$. This can be done most directly by using (7) of the proof, from which we get at once for a Bell quadruple (X_i, X_j, X_k, X_ℓ), with $1 \leq i < j \leq i_0 < k < \ell \leq N$ or $1 \leq k < \ell \leq i_0 < i < j \leq N$,

$$1 - |E(X_i X_k) - x_i x_j E(X_j X_k)| \geq x_i x_j E(X_i X_j) \qquad (11)$$

and

$$1 - |E(X_i X_\ell) - x_i x_j E(X_j X_\ell)| \geq x_i x_j E(X_i X_j). \qquad (12)$$

Setting $x_i x_j = 1$ in (11) and $x_i x_j = -1$ in (12) and adding the two inequalities, we get

$$|E(X_i X_k) - E(X_j X_k)| + |E(X_i X_\ell) + E(X_j X_\ell)| \leq 2,$$

a standard form of the Bell inequalities, with obvious permutation of the indices k and ℓ permitted.

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