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Probabilistic Results for Six Detectors in a Three-Particle GHZ Experiment

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Abstract. In this paper we show that the GHZ theorem can be reformulated as a probablistic theorem allowing for inefficiencies in the detectors. We show quantitatively that taking into account these inefficiencies, the published results of the Innsbruck experiment support the nonexistence of a joint probability distribution for the six correlated spin variables, and hence the nonexistence of hidden variables that explain the experimental results.

1 Introduction

One of the most influential papers on the foundations of quantum mechanics is undoubtedly that of Einstein, Podolski and Rosen (EPR), where they analyzed the correlations between observables of entangled two-particle states [1]. In their paper, EPR conclude that, because entangled systems can be perfectly correlated, Bohr's interpretation of quantum mechanics was incomplete, and a new theory using hidden variables was necessary. However, in 1963 John Bell showed that if we wanted a hidden-variable theory for quantum mechanics, as EPR did, this theory would be falsifiable, as it would yield correlations incompatible with the ones predicted by quantum mechanics [2]. In 1982, in a famous experiment, Aspect, Dalibard and Roger showed that quantum mechanical correlations were satisfied, in contradiction to the local hidden-variable predictions [3].

Recently, Greenberger, Horne and Zeilinger (GHZ) proposed another test for quantum mechanics against hidden-variable theories [4]. What makes the GHZ proposal distinct from Bell's inequalities is the fact that, by using the quantum mechanical predicted correlations for entangled states of more than two particles, they could obtain a mathematical contradiction if a local hidden-variable theory were assumed. The contradiction obtained for the case of three particles, as derived by Mermin in [5], goes as follows. Let $|\psi\rangle$ be a three-particle entangled spin state defined by

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+++\rangle + |---\rangle), \tag{1}$$

where we use the notation $|+++\rangle = |+\rangle_1 \otimes |+\rangle_2 \otimes |+\rangle_3$, and $\hat{\sigma}_{1z}|+\rangle_1 = |+\rangle_1$, $\hat{\sigma}_{1z}|-\rangle_1 = -|-\rangle_1$, $\hat{\sigma}_{1z}$ being the spin observable in the z direction acting on particle 1, and so on. The wavefunction (1) is an eigenstate of the following operators:

$$\hat{A}|\psi\rangle = \hat{\sigma}_{1x}\hat{\sigma}_{2y}\hat{\sigma}_{3y}|\psi\rangle = |\psi\rangle, \quad \hat{B}|\psi\rangle = \hat{\sigma}_{1y}\hat{\sigma}_{2x}\hat{\sigma}_{3y}|\psi\rangle = |\psi\rangle, \tag{2}$$

$$\hat{C}|\psi\rangle = \hat{\sigma}_{1y}\hat{\sigma}_{2y}\hat{\sigma}_{3x}|\psi\rangle = |\psi\rangle, \quad \hat{D}|\psi\rangle = \hat{\sigma}_{1x}\hat{\sigma}_{2x}\hat{\sigma}_{3x}|\psi\rangle = -|\psi\rangle. \tag{3}$$

From equations (2)–(3) we have at once that

$$E(\hat{A}) = E(\hat{B}) = E(\hat{C}) = 1 \tag{4}$$

and

$$E(\hat{D}) = -1. \tag{5}$$

To show the contradiction, we assume there is a hidden variable λ, that is a function of the source that generated the entangled state and that determines the value of each ot the components of spin for the particles. Let us denote by $m_{ix}(\lambda)$ the value ± 1 of the spin $\hat{\sigma}_{ix}$ in the x direction for particle i, $i = 1, 2, 3$, and by $m_{iy}(\lambda)$ the value of the spin $\hat{\sigma}_{iy}$ in the y direction. Since the expectations are 1, we can write, from (2)–(3), that

$$m_{1x}(\lambda)m_{2y}(\lambda)m_{3y}(\lambda) = 1, \tag{6}$$

$$m_{1y}(\lambda)m_{2x}(\lambda)m_{3y}(\lambda) = 1, \tag{7}$$

$$m_{1y}(\lambda)m_{2y}(\lambda)m_{3x}(\lambda) = 1. \tag{8}$$

But

$$\begin{aligned} [m_{1x}m_{2y}m_{3y}]\,[m_{1y}m_{2x}m_{3y}]\,[m_{1y}m_{2y}m_{3x}] &= [m_{1x}m_{2x}m_{3x}]\,[m_{1y}^2 m_{2x}^2 m_{3y}^2] \\ &= [m_{1x}m_{2x}m_{3x}]\,, \end{aligned} \tag{9}$$

where we dropped the λ to simplify the notation. Equations (6)–(9) imply that

$$E(\hat{D}) = m_{1x}m_{2x}m_{3x} = 1,$$

in clear contradiction with the quantum mechanical prediction given by (5), thus showing that noncontextual hidden-variables are incompatible with the quantum mechanical predictions. Hence, quantum mechanics is not compatible with the completeness requirements of EPR. We must stress at this point that the proof of the GHZ theorem relies on the fact that, in (9), all three terms on the left-hand side of the equation have the same hidden variable. However, if we allow the hidden variable to be contextual, i.e., for its value to depend on the experimental setup given by the observable operators, then no contradiction can be derived, as can be shown by proving that a nonmonotonic upper probability distribution exists that is compatible with the quantum mechanical expectations [6].

The main characteristic of GHZ's proof, however, has a major problem. How can we verify experimentally predictions based on correlation-one statements, since experimentally one cannot obtain events perfectly correlated? This problem was also present in Bell's original paper, where he considered cases where the correlations were one. To "avoid Bell's experimentally unrealistic restrictions", Clauser, Horne, Shimony and Holt [7] derived a new set of inequalities that would take into account imperfections in the measurement process. However, Bell's inequalities are quite different from the GHZ case, where in the latter it is *necessary* to have experimentally unrealistic perfect correlations to justify the proof leading to a contradiction. This can be seen from the following theorem (a simplified version of the theorem found in [8]), where the previous proof is reduced to pure probabilistic arguments.

Theorem 1 Let $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$, $\mathbf{Y}_1$, $\mathbf{Y}_2$ and $\mathbf{Y}_3$ be six ± 1 random variables, representing the outcome of spin measurements in the x and y directions for particles $1, 2$, and 3, and let

(i) $E(\mathbf{X}_1\mathbf{Y}_2\mathbf{Y}_3) = E(\mathbf{Y}_1\mathbf{X}_2\mathbf{Y}_3) = E(\mathbf{Y}_1\mathbf{Y}_2\mathbf{X}_3) = 1$,

(ii) $E(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3) = -1$.

Then (i)–(ii) imply a contradiction.

In a previous paper [9] we analyzed the Innsbruck three-particle GHZ experiment [10], where a GHZ state was generated, and showed that their experiment was not compatible with a noncontextual hidden-variable theory. To do this, we showed that a set of inequalities for three-particle correlations were necessary and sufficient for the existence of a noncontextual hidden-variable. However, our proof did not use the full set of six detection random variables $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$, $\mathbf{Y}_1$, $\mathbf{Y}_2$ and $\mathbf{Y}_3$, but instead used random variables defined as products of the $\mathbf{X}_i$'s and $\mathbf{Y}_i$'s, in a way similar to that used in the proof of Theorem 1. As a consequence, the set of inequalities presented in [9] is not symmetric in all random variables, as are Bell's inequalities. The present paper can be considered as a continuation of our early work [9], as we obtain conditions that, if verified experimentally, guarantee the nonexistence of a joint probability distribution for the particle observables $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$, $\mathbf{Y}_1$, $\mathbf{Y}_2$ and $\mathbf{Y}_3$, and hence the nonexistence of a hidden-variable [11]. The main difference from the inequalities obtained here and the ones from [9] is that, in the form presented below in Theorem 2, the inequalities assume a completely symmetric form. Also, since the publication of [9] new data appeared from the group in Innsbruck [12], and in Section 3 we refine the analysis made previously in the light of these new data, concluding once again that no noncontextual hidden variables are compatible with the outcomes of the experiment.

2 Bell-Like Inequalities for the GHZ State

In this Section we want to derive inequalities that guarantee the nonexistence of hidden-variables for the observables used in GHZ's proof. Before we derive those inequalities, it is important to note that if we could measure all the random variables $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$, $\mathbf{Y}_1$, $\mathbf{Y}_2$ and $\mathbf{Y}_3$ of Theorem 1 simultaneously, we would have a joint probability distribution. The existence of a joint probability distribution is a necessary and sufficient condition for the existence of a noncontextual hidden variable [11]. Hence, if the quantum mechanical GHZ correlations are obtained, then no such hidden variable exists. However, Theorem 1 still involves probability-one statements. On the other hand, the quantum mechanical correlations present in the GHZ state are so strong that even if we allow for experimental errors, the nonexistence of a joint distribution can still be verified, as we show in the following theorem, which, as we said above, extends the results in [9].

Theorem 2 Let X_i and Y_i, $1 \leq i \leq 3$, be six ± 1 random variables. Then, there exists a joint probability distribution for all six random variables if and only if the following inequalities are satisfied:
$-2 \leq E(X_1Y_2Y_3) + E(Y_1X_2Y_3) + E(Y_1Y_2X_3) - E(X_1X_2X_3) \leq 2,$
$-2 \leq -E(X_1Y_2Y_3) + E(Y_1X_2Y_3) + E(Y_1Y_2X_3) + E(X_1X_2X_3) \leq 2,$
$-2 \leq E(X_1Y_2Y_3) - E(Y_1X_2Y_3) + E(Y_1Y_2X_3) + E(X_1X_2X_3) \leq 2,$
$-2 \leq E(X_1Y_2Y_3) + E(Y_1X_2Y_3) - E(Y_1Y_2X_3) + E(X_1X_2X_3) \leq 2.$

Proof: The argument is similar to the one found in [9]. To simplify, we use a notation where $x_1y_2y_3$ means $X_1Y_2Y_3 = 1$, $\overline{x_1y_2y_3}$ means $X_1Y_2Y_3 = -1$. We prove first that the existence of a joint probability distribution implies the four inequalities. Then, we have by an elementary probability computation that

$$\begin{aligned} P(x_1y_2y_3) = {} & P(x_1y_2y_3, y_1x_2y_3, y_1y_2x_3) + P(x_1y_2y_3, \overline{y_1x_2y_3}, y_1y_2x_3) \\ & + P(x_1y_2y_3, y_1x_2y_3, \overline{y_1y_2x_3}) + P(x_1y_2y_3, \overline{y_1x_2y_3}, \overline{y_1y_2x_3}) \end{aligned}$$

and

$$\begin{aligned} P(\overline{x_1y_2y_3}) = {} & P(\overline{x_1y_2y_3}, y_1x_2y_3, y_1y_2x_3) + P(\overline{x_1y_2y_3}, \overline{y_1x_2y_3}, y_1y_2x_3) \\ & + P(\overline{x_1y_2y_3}, y_1x_2y_3, \overline{y_1y_2x_3}) + P(\overline{x_1y_2y_3}, \overline{y_1x_2y_3}, \overline{y_1y_2x_3}), \end{aligned}$$

with similar equations for $Y_1X_2Y_3$ and $Y_1Y_2X_3$. But

$$X_1X_2X_3 = (X_1Y_2Y_3)(Y_1X_2Y_3)(Y_1Y_2X_3),$$

and so we have that

$$\begin{aligned} P(x_1x_2x_3) = {} & P(x_1y_2y_3, y_1x_2y_3, y_1y_2x_3) + P(\overline{x_1y_2y_3}, \overline{y_1x_2y_3}, y_1y_2x_3) \\ & + P(x_1y_2y_3, \overline{y_1x_2y_3}, \overline{y_1y_2x_3}) + P(\overline{x_1y_2y_3}, y_1x_2y_3, \overline{y_1y_2x_3}) \end{aligned}$$

and

$$\begin{aligned} P(\overline{x_1x_2x_3}) = {} & P(\overline{x_1y_2y_3}, \overline{y_1x_2y_3}, \overline{y_1y_2x_3}) + P(\overline{x_1y_2y_3}, y_1x_2y_3, y_1y_2x_3) \\ & + P(x_1y_2y_3, \overline{y_1x_2y_3}, y_1y_2x_3) + P(x_1y_2y_3, y_1x_2y_3, \overline{y_1y_2x_3}). \end{aligned}$$

A straightforward computation shows that

$$\begin{aligned} F = {} & 2[P(x_1y_2y_3, y_1x_2y_3, y_1y_2x_3) + P(\overline{x_1y_2y_3}, y_1x_2y_3, y_1y_2x_3) \\ & + P(x_1y_2y_3, \overline{y_1x_2y_3}, y_1y_2x_3) + P(x_1y_2y_3, y_1x_2y_3, \overline{y_1y_2x_3})] \\ & - 2[P(\overline{x_1y_2y_3}, \overline{y_1x_2y_3}, \overline{y_1y_2x_3}) + P(\overline{x_1y_2y_3}, \overline{y_1x_2y_3}, y_1y_2x_3) \\ & + P(\overline{x_1y_2y_3}, y_1x_2y_3, \overline{y_1y_2x_3}) + P(x_1y_2y_3, \overline{y_1x_2y_3}, \overline{y_1y_2x_3})], \end{aligned}$$

where F is defined by

$$F = E(\mathbf{X}_1\mathbf{Y}_2\mathbf{Y}_3) + E(\mathbf{Y}_1\mathbf{X}_2\mathbf{Y}_3) + E(\mathbf{Y}_1\mathbf{Y}_2\mathbf{X}_3) - E(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3).$$

Since all probabilities are non-negative and sum to ≤ 1, we infer the first inequality at once. The derivation of the other inequalities is similar.

Now for the sufficiency part. First, we assume the symmetric case where

$$E(\mathbf{X}_1\mathbf{Y}_2\mathbf{Y}_3) = E(\mathbf{Y}_1\mathbf{X}_2\mathbf{Y}_3) = E(\mathbf{Y}_1\mathbf{Y}_2\mathbf{X}_3) = 2p - 1, \tag{10}$$

and

$$E(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3) = -(2p-1). \tag{11}$$

Then, the first inequality yields

$$\frac{1}{4} \leq p \leq \frac{3}{4}, \tag{12}$$

while the other ones yield

$$0 \leq p \leq 1. \tag{13}$$

Since $\mathbf{X}_i$ and $\mathbf{Y}_i$ are ± 1 random variables, p has to belong to the interval $[0,1]$, and inequality (13) doesn't add anything new. We will prove the existence of a joint probability distribution for this symmetric case by showing that, given any p, $\frac{1}{4} \leq p \leq \frac{3}{4}$, we can assign values to the atoms that have the proper marginal distributions.

The probability space for $\mathbf{X}_i$ and $\mathbf{Y}_i$ has 64 atoms. It is difficult to handle a problem of this size, so we will assume some further symmetries to reduce the number of atoms. We will first introduce the following notation: if a group of symbols is between square brackets, all the possible permutations of the barred symbol will considered. For example, we assume $a_5 = P([\bar{x}_1x_2x_3]y_1y_2y_3)$, meaning that $P(\bar{x}_1x_2x_3y_1y_2y_3) = a_5$, $P(x_1\bar{x}_2x_3y_1y_2y_3) = a_5$, and $P(x_1x_2\bar{x}_3y_1y_2y_3) = a_5$. Then, the number of atoms in the problem is reduced to the following 16: $a_1 = P(x_1x_2x_3y_1y_2y_3)$, $a_2 = P(x_1x_2x_3\bar{y}_1\bar{y}_2\bar{y}_3)$, $a_3 = P(x_1x_2x_3[\bar{y}_1y_2y_3])$, $a_4 = P(x_1x_2x_3[\bar{y}_1\bar{y}_2y_3])$, $a_5 = P([\bar{x}_1x_2x_3]y_1y_2y_3)$, $a_6 = P([\bar{x}_1x_2x_3]\bar{y}_1\bar{y}_2\bar{y}_3)$, $a_7 = P([\bar{x}_1x_2x_3][\bar{y}_1y_2y_3])$, $a_8 = P([\bar{x}_1x_2x_3][\bar{y}_1\bar{y}_2y_3])$, $a_9 = P([\bar{x}_1\bar{x}_2x_3][\bar{y}_1y_2y_3])$, $a_{10} = P([\bar{x}_1\bar{x}_2x_3][\bar{y}_1\bar{y}_2y_3])$, $a_{11} = P([\bar{x}_1\bar{x}_2x_3]y_1y_2y_3)$, $a_{12} = P([\bar{x}_1\bar{x}_2x_3]\bar{y}_1\bar{y}_2\bar{y}_3)$, $a_{13} = P(\bar{x}_1\bar{x}_2\bar{x}_3[\bar{y}_1y_2y_3])$, $a_{14} = P(\bar{x}_1\bar{x}_2\bar{x}_3[\bar{y}_1\bar{y}_2y_3])$, $a_{15} = P(\bar{x}_1\bar{x}_2\bar{x}_3y_1y_2y_3)$, $a_{16} = P(\bar{x}_1\bar{x}_2\bar{x}_3\bar{y}_1\bar{y}_2\bar{y}_3)$.

These new added symmetries reduce the problem from 64 to 16 variables. The atoms have to satisfy various sets of equations. The first set comes just from the requirement that $E(\mathbf{X}_i) = E(\mathbf{Y}_i) = 0$, for $i = 1,2,3$, but two of the six equations are redundant, and so we are left with the following four.

$$\begin{aligned} &a_1 + a_2 + 3a_3 + 3a_4 + a_5 + a_6 + 3a_7 + 3a_8 - 3a_9 \\ &-3a_{10} - a_{11} - a_{12} - 3a_{13} - 3a_{14} - a_{15} - a_{16} \quad = 0, \end{aligned} \tag{14}$$

$$\begin{aligned} &a_1 - a_2 + a_3 - a_4 + 3a_5 - 3a_6 + 3a_7 - 3a_8 + 3a_9 \\ &-3a_{10} + 3a_{11} - 3a_{12} + a_{13} - a_{14} + a_{15} - a_{16} \quad = 0, \end{aligned} \tag{15}$$

$$\begin{aligned} &a_1 - a_2 + a_3 - a_4 + 3a_5 - 3a_6 + 3a_7 - 3a_8 + 3a_9 \\ &-3a_{10} + 3a_{11} - 3a_{12} - a_{13} + a_{14} + a_{15} - a_{16} \qquad = 0, \end{aligned} \tag{16}$$

$$\begin{aligned} &a_1 + a_2 + 3a_3 + 3a_4 - a_5 - a_6 - 3a_7 - 3a_8 + 3a_9 \\ &+3a_{10} + a_{11} + a_{12} - 3a_{13} - 3a_{14} - a_{15} - a_{16} \qquad = 0, \end{aligned} \tag{17}$$

where (14) comes from $E(\mathbf{X}_1) = 0$, (15) from $E(\mathbf{X}_2) = 0$, (16) from $E(\mathbf{Y}_1) = 0$, and (17) from $E(\mathbf{Y}_2) = 0$. The triple expectations also imply

$$\begin{aligned} &a_1 + a_2 + 3a_3 + 3a_4 - 3a_5 - 3a_6 - 9a_7 - 9a_8 + 9a_9 \\ &+9a_{10} + 3a_{11} + 3a_{12} - 3a_{13} - 3a_{14} - a_{15} - a_{16} \qquad = -2p + 1, \end{aligned} \tag{18}$$

$$\begin{aligned} &a_1 + a_2 - a_4 - a_4 + a_5 + a_6 - a_7 - a_8 + a_9 \\ &+a_{10} - a_{11} - a_{12} + a_{13} + a_{14} - a_{15} - a_{16} \quad = 2p - 1, \end{aligned} \tag{19}$$

and

$$\begin{aligned} &a_1 - a_2 - 3a_3 + 3a_4 + 3a_5 - 3a_6 - 9a_7 + 9a_8 - 9a_9 \\ &+9a_{10} + 3a_{11} - 3a_{12} - 3a_{13} + 3a_{14} + a_{15} - a_{16} \qquad = 2p - 1. \end{aligned} \tag{20}$$

Finally, the probabilities of all atoms have to sum to one, yielding the last equation

$$\begin{aligned} &a_1 + a_2 + 3a_3 + 3a_4 + 3a_5 + 3a_6 + 9a_7 + 9a_8 + 9a_9 \\ &+9a_{10} + 3a_{11} + 3a_{12} + 3a_{13} + 3a_{14} + a_{15} + a_{16} \qquad = 1. \end{aligned} \tag{21}$$

Even with the symmetries reducing the problem to 16 variables, we still have an infinite number of solutions that satisfy Eqns. (14)–(21). Since it is very hard to exhibit a general solution for (14)–(21) with the constraints $0 \le a_i \le 1$, $i = 1 \dots 16$, we will just show that particular solutions exist for an arbitrary p satisfying the inequality (12). To do so, we will divide the problem into two parts: one where we will exhibit an explicit solution for the atoms $a_1, \dots, a_{16}$ that form a proper probability distribution for $p \in [\frac{1}{4}, \frac{1}{2}]$, and another explicit solution for $p \in [\frac{1}{2}, \frac{3}{4}]$.

It is easy to verify that, given an arbitrary p in $[\frac{1}{4}, \frac{1}{2}]$, the following set of values constitute a solution of Eqns. (14)–(21): $a_1 = 0$, $a_2 = -\frac{1}{2}+2p$, $a_3 = \frac{1}{4}-\frac{1}{2}p$, $a_4 = 0$, $a_5 = 0$, $a_6 = 0$, $a_7 = 0$, $a_8 = 0$, $a_9 = 0$, $a_{10} = 0$, $a_{11} = 0$, $a_{12} = \frac{1}{4} - \frac{1}{2}p$, $a_{13} = 0$, $a_{14} = 0$, $a_{15} = p$, $a_{16} = 0$. For p in $[\frac{1}{2}, \frac{3}{4}]$ the following set of values constitute a solution of Eqns. (14)–(21): $a_1 = -\frac{1}{8} + \frac{1}{2}p$, $a_2 = 0$, $a_3 = \frac{3}{8} - \frac{1}{2}p$, $a_4 = 0$, $a_5 = -\frac{5}{24} + \frac{1}{3}p$, $a_6 = -\frac{1}{24} + \frac{1}{6}p$, $a_7 = 0$, $a_8 = 0$, $a_9 = 0$, $a_{10} = 0$, $a_{11} = 0$, $a_{12} = 0$, $a_{13} = 0$, $a_{14} = \frac{1}{8}$, $a_{15} = \frac{3}{8} - \frac{1}{2}p$, $a_{16} = 0$. So, for p satisfying the inequality $\frac{1}{4} \le p \le \frac{3}{4}$ we can always construct a probability distribution for the atoms consistent with the marginals, and this concludes the proof. $\diamondsuit$

We note that the form of the inequalities of Theorem 2 is actually that of the Clauser et al. inequalities [7] for the Bell case, when the Bell binary correlations are replaced by the GHZ triple correlations. The inequalities from Theorem 2 immediately yield the following.

Corollary Let $\mathbf{X}_i$ and $\mathbf{Y}_i$, $1 \leq i \leq 3$, be six ± 1 random variables, and let
(i) $E(\mathbf{X}_1\mathbf{Y}_2\mathbf{Y}_3) = E(\mathbf{Y}_1\mathbf{X}_2\mathbf{Y}_3) = E(\mathbf{Y}_1\mathbf{Y}_2\mathbf{X}_3) = 1 - \varepsilon$,
(ii) $E(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3) = -1 + \varepsilon$, with $\varepsilon \in [0, 1]$.
Then there cannot exist a joint probability distribution of $\mathbf{X}_i$ and $\mathbf{Y}_i$, $1 \leq i \leq 3$, satisfying (i) and (ii) if $\varepsilon < \frac{1}{2}$.

Proof. If a joint probability exists, then

$$-2 \leq E(\mathbf{X}_1\mathbf{Y}_2\mathbf{Y}_3) + E(\mathbf{Y}_1\mathbf{X}_2\mathbf{Y}_3) + E(\mathbf{Y}_1\mathbf{Y}_2\mathbf{X}_3) - E(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3) \leq 2.$$

But

$$E(\mathbf{X}_1\mathbf{Y}_2\mathbf{Y}_3) + E(\mathbf{Y}_1\mathbf{X}_2\mathbf{Y}_3) + E(\mathbf{Y}_1\mathbf{Y}_2\mathbf{X}_3) - E(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3) = 4 - 4\varepsilon,$$

and the inequality is satisfied only if $\varepsilon \geq \frac{1}{2}$. Hence, if $\varepsilon < \frac{1}{2}$ no joint probability exists. ◇

In the Corollary, ε may represent, for example, a deviation, due to experimental errors, from the predicted quantum mechanical correlations. So, we see that to prove the nonexistence of a joint probability distribution for the GHZ experiment, we do not need to have perfect measurements and 1 or -1 correlations. In fact, from the inequality obtained above, it is clear that any experiment that satisfies the strong symmetry of the Corollary and obtains a correlation for the triples stronger than 0.5 (and -0.5 for one of them) cannot have a joint probability distribution. It is worth mentioning at this point that the inequalities derived in Theorem 2 have a completely different origin than do Bell's inequalities. The inequalities of Theorem 2 are not satisfied by a particular model, but they just accommodate the theoretical conditions in GHZ to possible experimental deviations. Also, Theorem 2 does not rely on any "enhancement" hypothesis to reach its conclusion. Thus, with this reformulation of the GHZ theorem it is possible to use strong, yet imperfect, experimental correlations to prove that a hidden-variable theory is incompatible with the experimental results.

To complete this Section, we will now present a result that is similar to one obtained by Garg and Mermin for the case of Bell's inequalities [13]. Garg and Mermin constructed a set of eight ± 1 random variables that satisfy Bell's inequalities, but do not have a joint probability distribution. For our example, it is sufficient to increase the number of random variables from six to seven.

Theorem 3 Let $\mathbf{X}_i$, $\mathbf{Y}_i$, and $\mathbf{Y}_1'$, $1 \leq i \leq 3$, be seven ± 1 random variables with expectations equal to zero. The satisfaction of the inequalities of Theorem 2 by the subsets of random variables $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3\}$ and $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{Y}_1', \mathbf{Y}_2, \mathbf{Y}_3\}$ does not guarantee the existence of a joint probability distribution for the full set of random variables.

Proof. We prove Theorem 3 by constructing an example of seven variables not having a joint probability distribution. Say we have the following expectations, all satisfying the inequalities of Theorem 2:

$$E(\mathbf{Y}_1\mathbf{X}_2\mathbf{X}_3) = E(\mathbf{X}_1\mathbf{Y}_2\mathbf{X}_3) = E(\mathbf{X}_1\mathbf{X}_2\mathbf{Y}_3) = E(\mathbf{Y}_1\mathbf{Y}_2\mathbf{Y}_3) = -1, \qquad (22)$$

$$E(\mathrm{Y}_1^{'}\mathrm{X}_2\mathrm{X}_3) = E(\mathrm{Y}_1^{'}\mathrm{Y}_2\mathrm{Y}_3) = 0, \;\; E(\mathrm{X}_1\mathrm{Y}_2\mathrm{X}_3) = E(\mathrm{X}_1\mathrm{X}_2\mathrm{Y}_3) = -1. \quad (23)$$

Since $E(\mathrm{Y}_1^{'}\mathrm{X}_2\mathrm{X}_3) = E(\mathrm{Y}_1^{'}\mathrm{Y}_2\mathrm{Y}_3) = 0$, a joint probability for the six variables of (23) exists with the correlations

$$E(\mathrm{Y}_1^{'}\mathrm{Y}_2) = 1, \quad E(\mathrm{Y}_1^{'}\mathrm{X}_2) = 1, \quad E(\mathrm{Y}_1^{'}\mathrm{Y}_3) = 1, \quad E(\mathrm{Y}_1^{'}\mathrm{X}_3) = -1.$$

From the correlations fixed above, we can obtain immediately that

$$E(\mathrm{X}_2\mathrm{Y}_2) = 1, \quad E(\mathrm{X}_3\mathrm{Y}_3) = -1.$$

But, we have also that

$$E(\mathrm{Y}_1\mathrm{Y}_2\mathrm{Y}_3) = E((\mathrm{Y}_1\mathrm{X}_2\mathrm{X}_3)(\mathrm{X}_2\mathrm{Y}_2)(\mathrm{X}_3\mathrm{Y}_3)) = -E(\mathrm{Y}_1\mathrm{X}_2\mathrm{X}_3),$$

thus showing that the assumption of a joint probability distribution leads to a contradiction with (22). ◇

The counterexample to the existence of a joint probability distribution just constructed, and also the one constructed by Garg and Mermin for Bell's inequalities, simply illustrate a well-known feature of probability theory that has nothing to do with quantum mechanics. The feature is this. Given marginal probability distributions for pairs, triples, or, in general, n-tuples, $n = 1, 2, \ldots$, of random variables, there will not exist a joint probability distribution for m-tuples of random variables, $m > n$, without making special assumptions. Of course, distributions satisfying special assumptions are of great importance, e.g., the Gaussian or normal distribution of n random variables for which only means, variances and covariances are needed to fix the distribution uniquely. Consequently, for a Gaussian, all higher moments are already fixed by the first and second moments.

It is also to be emphasized that giving inequalities for larger numbers of random variables, either in the Bell or GHZ case, necessary and sufficient for the existence of a joint probability distribution is difficult, and very likely intractable for arbitrary n. For a sense of the difficulties in the Bell case, see Suppes and Zanotti [11].

3 Analysis of the Innsbruck Experiment

In this Section, our results from [9] are applied to the most recent experimental results of the Innsbruck group [10,12]. Before we proceed, we need to lay down the experimental setup that we consider. A schematic of the Innsbruck experiment is shown in Figure 1. In this experiment an UV laser pulse is sent into a nonlinear crystal, generating correlated pairs of photons, that can be detected in four detectors T, D_1, D_2, and D_3. In the Figure, "Pol." are polarizers, "BS" is a 50-50 beam splitter, and $\lambda/2$ is a half-wave plate. There is a small, yet nonzero, probability that two pairs of photons are generated simultaneously, where by simultaneous we mean within a window of observation Δt. It can be shown theoretically that, for the case of two-pair production, when one of the photons is detected in the trigger detector T, the other three photons detected in D_1,

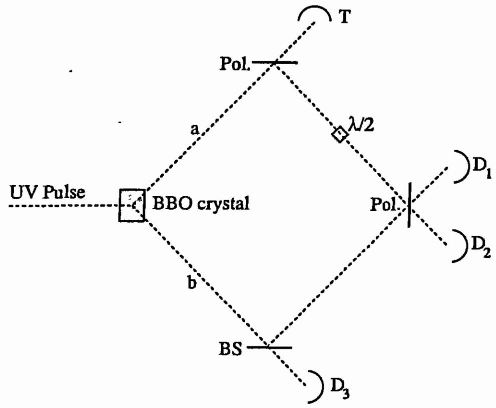

Fig. 1. Schematics of the GHZ experiment of Bouwmeester et al.

D_2, and D_3 are in a GHZ entangled state. In other words, if we conditionalize the experimental data to a detection at T, then D_1, D_2, and D_3 should show a GHZ correlation. There are several different sources of inefficiencies that we have taken into account in [9]: the detection efficiency, the dark-count rate, and the misfiring probability, just to name a few, but important, examples. We use the analysis from [9] of what happens to the GHZ correlations in a setup like Figure 1 when the detectors have efficiency $d \in [0,1]$ and a dark-count rate of probability γ. We assumed that the misfiring probabilities are negligible, as the polarizers used in the wavelengths in question are very efficient. The analysis also assumed that whenever a double pair is produced, this double pair has the expected GHZ correlation for D_1, D_2, and D_3 if T correctly detects a photon. The probability of no photon being generated and having a fourfold coincidence registered was considered negligible, as this would require a dark-count rate much bigger than found in modern photon detectors. Also, triple-pair creations were neglected because they are of extremely low probability compared to the other events. In [9] we obtained as the expression for the spin correlations the following.

$$E\left(\mathbf{S}_1\mathbf{S}_2\mathbf{S}_3 \mid td_1d_2d_3\right) = \frac{E\left(\mathbf{S}_1\mathbf{S}_2\mathbf{S}_3 \mid td_1d_2d_3 \ \&\ GHZ\right)}{\left[1 + 6\frac{P(p_1p_2)}{P(p_1 \ldots p_4)}\frac{\gamma^2}{d^2}\right]}. \tag{24}$$

This value is the corrected expression for the correlations if we have detector efficiency taken into account.

We estimate the values of γ and d to see how much $E\left(\mathbf{S}_1\mathbf{S}_2\mathbf{S}_3 \mid td_1d_2d_3\right)$ would change due to experimental errors. For that purpose, we will use typical rates of detectors [14] for the frequency used at the Innsbruck experiment, as well as their reported data [10]. First, modern detectors usually have $d \approx 0.5$ for the wavelengths used at Innsbruck. We assume a dark-count rate of about 3×10^2 counts/s. With a time window of coincidence measurement of 2×10^{-9} s, we then have that the probability of a dark count in this window is $\gamma = 6 \times 10^{-7}$. From [10] we use that the ratio $P(p_1p_2)/P(p_1 \ldots p_2)$ is on the order of 10^{10}. Substituting these numerical values in (24) we have $E\left(\mathbb{S}_1\mathbb{S}_2\mathbb{S}_3 \mid td_1d_2d_3\right) \cong 0.9$. From this expression it is clear that the change in correlation imposed by the dark-count rates is significant for the given parameters. However, it is also clear that the value of the correlation is quite sensitive to changes in the values of both γ and d.

We can now compare the values we obtained with the ones observed by Bouwmeester et al. for GHZ and $\overline{GHZ}$ states [12]. Using the notation from previous sections, they report that for the settings $\mathbf{Y}_1\mathbf{X}_2\mathbf{X}_3$, $\mathbf{X}_1\mathbf{Y}_2\mathbf{X}_3$, and $\mathbf{X}_1\mathbf{X}_2\mathbf{Y}_3$ a fraction of 0.85 ± 0.04 had the predicted quantum mechanical correlation, i.e.,

$$E(\mathbf{Y}_1\mathbf{X}_2\mathbf{X}_3) + E(\mathbf{X}_1\mathbf{Y}_2\mathbf{X}_3) + E(\mathbf{X}_1\mathbf{X}_2\mathbf{Y}_3) = 2.55.$$

However, for the experimental setting $\mathbf{Y}_1\mathbf{Y}_2\mathbf{Y}_3$ they report a correlation

$$E(\mathbf{Y}_1\mathbf{Y}_2\mathbf{Y}_3) = -0.74.$$

If we use the inequalities from Theorem 2, we see that the first inequality is clearly violated, as

$$E(\mathbf{Y}_1\mathbf{X}_2\mathbf{X}_3) + E(\mathbf{X}_1\mathbf{Y}_2\mathbf{X}_3) + E(\mathbf{X}_1\mathbf{X}_2\mathbf{Y}_3) - E(\mathbf{Y}_1\mathbf{Y}_2\mathbf{Y}_3) = 3.29 > 2.$$

Using Theorem 2, this result strongly supports the nonexistence of a joint probability distribution.

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