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QUANTIFIER-FREE AXIOMS FOR CONSTRUCTIVE AFFINE PLANE GEOMETRY

The purpose of this paper is to state a set of axioms for plane geometry which do not use any quantifiers, but only constructive operations. The relevant definitions and general theorems are stated; for reasons of space the proofs are only sketched. Quantifier-free arithmetic and quantifier-free algebra have been the subjects of several investigations, beginning at least with the early important work of Herbrand. Quantifier-free axioms for plane geometry have received less attention. In a way, this is surprising, for an emphasis on geometric constructions is a significant aspect of ancient Greek geometry.

The spirit of this article is that of Moler and Suppes (1968), who give quantifier-free axioms for the two constructions of finding the intersection of two lines and of laying off one segment on another. The representation theorem for models of their constructive theory is in terms of vector spaces over Pythagorean fields, a geometry discussed by Hilbert in his *Foundations of Geometry* but not axiomatized. The present work has two important differences. First, the emphasis is on finite configurations, rather than on closure of the constructions to yield a representation theorem in terms of a vector space over the ordered field of rationals. Second, the affine axioms, though numerous, are individually much simpler and avoid some troublesome problems corresponding to division by zero. The axioms are summarized without comment in the appendix. The axioms are clearly not independent; for example, those on linearity can be derived from the later order axioms.

1. PRIMITIVE CONCEPTS AND AXIOMS

The two affine binary operations taken as primitive are those of bisecting and doubling a line segment; the ternary relation of linearity is also taken as primitive. That these operations and relation are affine invariant is obvious, but it does not seem to have previously been shown that they are also sufficient for constructing finite affine planes – the exact sense of sufficiency will be commented on later. I use $\oplus$ for bisecting, $*$ for doubling, and L for

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linearity. Thus, $a \oplus b = c$ and $a * b = d$, as shown in Figure 1. Although line segments and even lines may be mentioned informally, as in Moler and Suppes (1968), only points are referred to in the axioms.

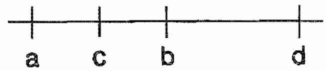

Figure 1.

To keep the axioms quantifier-free we postulate three points denoted α, β and γ, and the first axiom of collinearity is that these three points are not collinear. The other two axioms of collinearity are familiar in many geometric contexts.

(L1) It is not the case $L(\alpha\beta\gamma)$.

(L2) If $a = b, a = c$ or $b = c$, then $L(abc)$.

(L3) If $a \neq b$, $L(abp)$, $L(abq)$ and $L(abr)$ then $L(pqr)$.

Axiom (L1) makes the dimension of the space at least two, (L2) expresses a kind of ternary reflexivity and (L3) a kind of ternary transitivity.

The axioms on bisection are (B1)–(B4), for any points a, b, c and d:

(B1) $a \oplus a = a$ (Idempotency).

(B2) $a \oplus b = b \oplus a$ (Commutativity).

(B3) $(a \oplus b) \oplus (c \oplus d) = (a \oplus c) \oplus (b \oplus d)$ (Bicommutativity).

(B4) If $a \oplus b = a \oplus b'$ then $b = b'$ (Cancellation).

The axioms on doubling are (D1), (D2) and (D3) for any points a, a', b and b':

(D1) If $a * b = b * a$ then $a = b$ (Antisymmetry).

(D2) If $a * b = a * b'$ then $b = b'$ (Left Cancellation).

(D3) If $a * b = a' * b$ then $a = a'$ (Right Cancellation).

As can be seen from Figure 1, the doubling operation is not commutative when $a \neq b$. Finally, there is the axiom of reduction using both operations, the axiom expressing the linearity of bisection, and an Axiom (LL) that linearity of three midpoints implies linearity of the three points.

This last axiom may be provable by using the later axioms of order, but I have not been able to derive it.

(BD) $a \oplus (a * b) = b$ (Reduction).

(LB) $L(ab(a \oplus b))$ (Linearity of Bisection).

(LL) If $L((a \oplus b)(b \oplus c)(a \oplus c))$ then $L(abc)$.

Axioms (L2), (L3) and (B1)–(B3) are used by Szmielew (1983) in her development of midpoint algebras as a basis for affine geometry. Her approach is not constructive and so her development is different from that given here. In particular, she uses the following strong solvability axiom for bisection (which she calls the *midpoint* operation): For every a and b there exists an x such that $x \oplus a = b$. With only quantifier-free axioms, such a solvability condition cannot be formulated, and so something like the second operation of doubling introduced here is required.

2. THEOREMS

First, I summarize in one theorem the elementary properties of collinearity.

THEOREM 1. (Collinearity).

(i) If $L(abc)$ then L holds for any permutation of abc.
(ii) $L(aba)$.
(iii) If $a \neq b$, $L(abc)$ and $L(abd)$ then $L(acd)$.
(iv) If $p \neq q$, $L(abp)$, $L(abq)$ and $L(pqr)$ then $L(abr)$.

Szmielew (1983) points out that (i), (ii) and (iii) of Theorem 1 are equivalent to Axioms (L2) and (L3) – actually a weaker form of (i), namely, if $L(abc)$ then $L(bac)$.

THEOREM 2. $(a \oplus b) \oplus c = (a \oplus c) \oplus (b \oplus c)$ (Self-distributivity).

Proof. Using idempotency (B1) and bicommutativity (B3) $(a \oplus b) \oplus c = (a \oplus b) \oplus (c \oplus c) = (a \oplus c) \oplus (b \oplus c)$.

THEOREM 3. If $a \oplus b = a$ then $a = b$.

Proof. Using idempotency, $a \oplus b = a = a \oplus a$, then using cancellation, $a = b$.

The next theorem shows that reduction and linearity hold for doubling.

THEOREM 4. $a * (a \oplus b) = b$ and $L(ab(a * b))$.

Proof. By Axiom (BD)

$$a \oplus (a * (a \oplus b)) = a \oplus b,$$

and so by cancellation

$$a * (a \oplus b) = b.$$

In Axiom (LB), putting $a * b$ for b we obtain at once:

$$L(a(a * b)(a \oplus (a * b))),$$

but using now Axiom BD, we get,

$$L(a(a * b)b),$$

and by Theorem 1(i), $L(ab(a * b))$.

Any three noncollinear points "form" a triangle, so we may define the ternary relation T of triangularity as the negation of L.

DEFINITION 1. $T(abc)$ iff it is not the case $L(abc)$.

In the next definition the quaternary relation P defined has the intuitive meaning that four points standing in this relation form a parallelogram (thus "P" for "parallelogram").

DEFINITION 2. $P(abcd)$ iff $T(abc)$ & $a \oplus c = b \oplus d$.

This definition characterizes parallelograms, which do the work "locally" of parallel lines in standard nonconstructive affine geometry. The triangularity or nonlinearity condition eliminates degeneracy. The important condition is that a convex quadrilateral $abcd$ (see Figure 2) is a parallelogram if and only if the midpoints of the diagonals ac and bd coincide, which is a familiar property of parallelograms, but also sufficient for the definition. Concave quadrilaterals violate the midpoint condition.

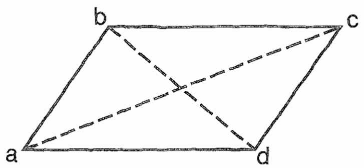

Figure 2

The following three theorems express elementary facts about parallelograms.

THEOREM 5. If $P(abcd)$ then it is not the case $P(acbd)$.

Proof. Suppose, by way of contradiction, that

$$(1) \qquad a \oplus b = c \oplus d,$$

and by hypothesis

$$(2) \qquad a \oplus c = b \oplus d.$$

Now by bicommutativity (B3)

$$(3) \qquad (a \oplus b) \oplus (c \oplus d) = (a \oplus c) \oplus (b \oplus d)$$

But by (1) and idempotency (B1)

$$(4) \qquad (a \oplus b) \oplus (c \oplus d) = (a \oplus b) \oplus (a \oplus b) = a \oplus b,$$

and similarly, using (2)

$$(5) \qquad (a \oplus c) \oplus (b \oplus d) = (a \oplus c) \oplus (a \oplus c) = a \oplus c.$$

From (3), (4) and (5) by cancellation (B4),

$$b = c,$$

which contradicts the hypothesis $T(abc)$.

The next theorem shows that three vertices of a parallelogram uniquely determine the fourth vertex.

THEOREM 6. If $P(abcd)$ and $P(abcd')$ then $d = d'$.

Proof. From the hypothesis we have the two midpoint identities

$$a \oplus b = c \oplus d \text{ and } a \oplus b = c \oplus d',$$

so immediately we infer

$$c \oplus d = c \oplus d',$$

and thus by cancellation

$$d = d'.$$

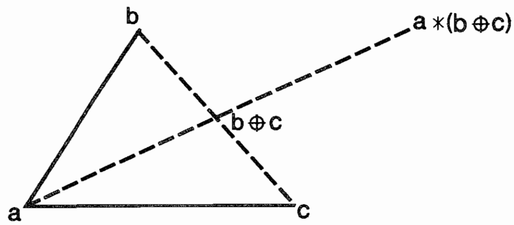

Figure 3.

The next theorem makes explicit the construction of the fourth vertex of a parallelogram, as can be seen from the construction shown in Figure 3.

THEOREM 7. If $T(abc)$ then $P(cab(a * (b \oplus c))$.

Proof. By hypothesis $T(abc)$ and so by Theorem 1(i) $T(cab)$.

Second, by Axiom (B2)

$$c \oplus b = b \oplus c,$$

and then from Axiom (BD) applied to the right-hand side

$$c \oplus b = a \oplus (a * (b \oplus c),$$

which is just the desired midpoint condition.

The next theorem shows that if one figure is a triangle, so is another constructed from the midpoints of two adjacent sides.

THEOREM 8. If $T(abc)$ then $T(a(a \oplus b)(a \oplus c))$.

Proof. By Axiom (LB)

(1) $L(ab(a \oplus b))$

and

(2) $L(ac(a \oplus c))$.

By way of contradiction, suppose

(3) $L(a(a \oplus b)(a \oplus c))$

using Theorem 1(i), we infer from (1)

(4) $L(a(a \oplus b)b)$

and then, since $a \neq b$ follows from the hypothesis $T(abc)$, by Theorem 3 $a \neq a \oplus b$, and so using Theorem 1(iii), we infer from (3) and (4)

(5) $L(a(a \oplus c)b)$,

and then, using Theorem 1(i), from (2), we have $L(a(a \oplus c)c)$, which together with (5) and use of Theorem 3 and Theorem 1(iii), yields

(6) $L(abc)$,

but (6) contradicts the hypothesis of the theorem, which completes the proof.

THEOREM 9. If $P(abcd)$ then $P(bcda)$, $P(cdab)$ and $P(dabc)$.

Proof. Since in all three cases the midpont condition $a \oplus c = b \oplus d$ on the diagonals follows at once from

(1) $a \oplus c = b \oplus d$

by commutativity of $\oplus$, it remains to prove $T(bcd)$, $T(cda)$ and $T(dab)$.

We prove $T(dab)$. The argument is similar for the other two cases.

First, we must show $d \neq a$ and $d \neq b$. If $d = a$, then from (1) $a \oplus c = b \oplus a$, and so $L(a(a \oplus b)(a \oplus c))$, which contradicts Theorem 8. If $d = b$, then by (1) $a \oplus c = b \oplus b = b$, and so from $L(a(a \oplus b)b)$ we infer again $L(a(a \oplus b)(a \oplus c))$, contradicting Theorem 8.

Now suppose, by way of contradiction,

(2) $L(abd)$

By linearity of $\oplus$

(3) $L(bd(b \oplus d))$

From (1) and (3)

(4) $L(bd(a \oplus c))$

and from (2)

(5) $L(bda)$.

From (4) and (5) by Theorem 1(iii), which requires $b \neq d$, and Theorem 1(i)

(6) $L(ab(a \oplus c))$,

and by linearity of $\oplus$ again

(7) $L(ab(a \oplus b))$,

which contradicts Theorem 8.

The next theorem is a "counterclockwise" version of Theorem 9 in terms of the vertices a, b, c, d of the given parallelogram. The proof is similar to that of Theorem 9.

THEOREM 10. If $P(abcd)$ then $P(adcb)$, $P(dcba)$, $P(cbad)$ and $P(badc)$.

Our next goal is to prove, in the present setup, the classical theorem that the segment joining the midpoints of two sides of a triangle is parallel to the third side. See Figure 4. Actually, by finding the midpoint of the third side, we get more, namely, the parallelogram described in the theorem.

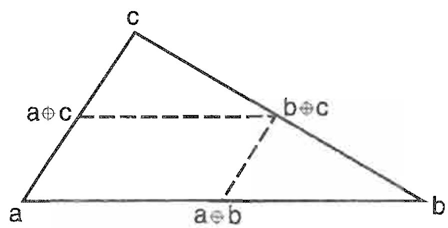

Figure 4

THEOREM 11. If $T(abc)$ then $P(a(a \oplus b)(b \oplus c)(a \oplus c))$.

Proof. By hypothesis $T(abc)$, so by the contrapositive of Axiom (LL) $T((a \oplus b)(b \oplus c)(a \oplus c))$.

Then the following identities prove $P((a \oplus b)(b \oplus c)(a \oplus c)a)$:

$$\begin{aligned} (a \oplus b) \oplus (a \oplus c) &= (b \oplus a) \oplus (c \oplus a) && \text{by (B2)},\\ &= (b \oplus c) \oplus a && \text{by Th. 2,} \end{aligned}$$

and using the third permutation of Theorem 9, we obtain the desired result.

The next theorem intuitively corresponds to the affine proposition that the binary relation of being parallel is transitive for segments or lines. Here it is formulated in terms of parallelograms, but the transitivity is evident, especially when Figure 5 is viewed along with the statement of the theorem. The hypothesis of the theorem has two triangular conditions $T(abe)$ and $T(abf)$ to exclude collinearity.

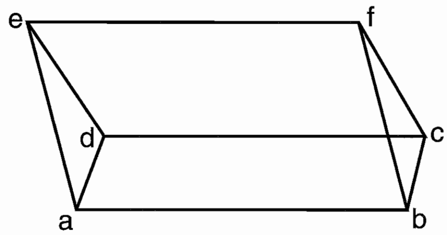

Figure 5.

THEOREM 12. If $P(abcd)$, $P(cdef)$, $T(abe)$ and $T(abf)$ then $P(abfe)$.

Proof. $T(abf)$ is part of the hypothesis, and $e \neq f$ follows from the assumption that $P(cdef)$. From $P(abcd)$ and $P(cdef)$, we have at once the two midpoint identities:

$$\begin{aligned} a \oplus c &= b \oplus d, \\ c \oplus e &= d \oplus f, \end{aligned}$$

from which we infer at once

$$(a \oplus c) \oplus (d \oplus f) = (b \oplus d) \oplus (c \oplus e).$$

Using repeatedly commutativity and bicommutativity, we get

$$(a \oplus f) \oplus (c \oplus d) = (b \oplus e) \oplus (c \oplus d),$$

and then by cancellation, the desired midpoint identity

$$a \oplus f = b \oplus e$$

follows.

The ease with which the analogues of classical geometric theorems were proved in the present affine setting shows how powerful and natural the geometry of midpoints or bisection is. The next three sections extend the results beyond purely geometric considerations to the introduction of coordinates, to affine order and then the separate foundational topic of constructing rational numbers.

3. INTRODUCING COORDINATES

First of all, rather than follow the approach in Moler and Suppes (1968) of proving the representation theorem for operations whose closure requires an infinite domain of points, it is, in my current view, more in the spirit of constructions to introduce coordinates for the finite configuration of points of individual constructions. From a formal standpoint, to avoid closure, I should replace the bisection operation $a \oplus b$ by a ternary relation $B(a, b, c)$ such that if $B(a, b, c)$ and $B(a, b, c')$ then $c = c'$. Similar remarks apply to the doubling operation $a * b$. But I shall not actually make this change of notation, but have it understood implicity in this section and the next.

I also define a *finite affine plane configuration* in intuitive language that is easily formalized:

(i) the set of points of the configuration is finite;
(ii) at most three arbitrary points are given;
(iii) if three arbitrary points are given they must be noncollinear;
(iv) only the operations of bisecting and doubling are used to construct points that are not given.

Given points α, β and γ such that $T(\alpha, \beta, \gamma)$, and thus they are noncollinear, we introduce coordinates as shown in Figure 6. Obviously it is

Figure 6

purely conventional how the coordinates are assigned to the three points – there are six possibilities. For convenience we use only the one shown in Figure 6. The numerical interpretation of the two operations of bisection and doubling are straightforward. If the coordinates of a and b are

$$a : (x, y) \text{ and } b : (u, v),$$

then the coordinates of $a \oplus b$ and $a * b$ are

$$a \oplus b : \quad \left(\frac{x+u}{2}, \frac{y+v}{2}\right),$$
$$a * b : (2u - x), (2v - y).$$

The numerical interpretation of the ternary collinear relation is standard. If $L(abc)$ then for some rational number t, the numerical coordinates (r, s) for c satisfy:

$$r = tx + (1 - t)u, \; s = ty + (1 - t)v.$$

We now state the following theorem about such coordinates, whose proof, though elementary, is too long to give here. For what follows in the next section we want to restrict the coordinates to rational numbers, which is the meaning in the statement of the theorem that φ_1 and φ_2 are rational functions.

THEOREM 13. Let $\mathcal{A} = (A, \oplus, *, \alpha, \beta, \gamma, L)$ *be a finite affine plane configuration. Then there is a unique pair* (φ_1, φ_2) of rational numerical functions defined on A such that for every a and b in A:

(i) $\varphi_1(\alpha) = 0$, $\varphi_2(\alpha) = 0$, $\varphi_1(\beta) = 1$, $\varphi_2(\beta) = 0$, $\varphi_1(\gamma) = 0$, $\varphi_2(\gamma) = 1$.
(ii) $\varphi_i(a \oplus b) = \frac{\varphi_i(a)+\varphi_i(b)}{2}$, $i = 1, 2$.
(iii) $\varphi_i(a * b) = 2\varphi_i(b) - \varphi_i(a)$, $i = 1, 2$.
(iv) If $L(abc)$ then for some rational number t, $\varphi_i(c) = t\varphi_i(a) + (1 - t)\varphi_i(b)$, $i = 1, 2$.

4. AFFINE ORDER AND CONGRUENCE

The standard axioms of affine order are in terms of the ternary relation of betweenness, written $a|b|c$ for "b is between a and c".

(Bt1) If $a|b|a$ then $a = b$. (Reflexivity)

(Bt2) If $a|b|c$ then $c|b|a$. (Symmetry)

(Bt3) If $a|b|c$ and $b|d|c$ then $a|b|d$ (Transitivity)

(Bt4) If $L(abc)$ then $a|b|c$ or $b|c|a$ or $c|a|b$. (Connectivity)

Note. If the domain has exactly four elements then the following axiom is needed: *if* $a|b|c$, $b|c|d$ and $b \neq c$, then $a|b|d$. Proof that this axiom is independent only in a four-element domain is due to Piesyk (1977).

Because elementary properties of betweenness are derived from these or closely related axioms in many places, no proofs are given here.

THEOREM 14.

(i) $a|a|a$.
(ii) $a|a|b$.
(iii) If $a|b|c$ and $a|c|d$ *then* $b|c|d$.
(iv) If $a|b|c$ and $a|c|b$ then $b = c$.
(v) If $a \neq b$, $a|b|c$ and $a|b|d$ then $a|c|d$ or $a|d|c$.

(vi) If $a \neq b$, $a|b|c$ and $a|b|d$ then $b|c|d$ or $b|d|c$.

To relate betweenness and bisection, we add:

Axiom (BB). $a|a \oplus b|b$.

We have at once

THEOREM 15. $a|b|a * b$.

Proof. In Axiom (BB) replace b by $a * b$ to obtain

$$a|a \oplus (a * b)|a * b,$$

and then recall from Axiom (BD): $b = a \oplus (a * b)$.

We also add an axiom relating betweenness and linearity.

Axiom (BL). If $a|b|c$ then $L(abc)$.

The affine congruence axioms needed consist of four general axioms and three with particular constructions.

(C1) If $aa \approx bc$ then $b = c$.

(C2) $ab \approx ba$.

(C3) If $ab \approx cd$ and $ab \approx ef$ then $cd \approx ef$.

(C4) If $a|b|c$, $a'|b'|c'$, $ab \approx a'b'$ and $bc \approx b'c'$ then $ac \approx a'c'$ (Additivity).

(C5) $a(a \oplus b) \approx b(a \oplus b)$.

(C6) $a * b \approx b * a$.

(C7) If $P(abcd)$ then $ab \approx cd$ and $bc \approx ad$.

Note that Axiom (C7) is required in order to have segments that are parallel but not collinear congruent. Axioms (C1)–(C4) are taken from Suppes, Krantz, Luce and Tversky (1989, 109). It is easy to show that Axioms (C1) and (C2) imply that congruence is an equivalence relation for segments.

Some further developments needed for the construction of rational numbers now follow. First, the concept of two line segments being parallel is defined, with collinearity not permitted as a degenerate case. Second, the primitive concepts are extended to the construction of trapezoids.

DEFINITION 3. $ab \parallel cd$ iff $T(abc)$, $c \neq d$, $P(ab(a * (b \oplus d))d)$ and $L(c(a * (b \oplus d))d)$.

The parallelogram is used in the definition as the "local" concept of parallel, but formally the definition could be simplified by replacing mention of the parallelogram, by using the triangle $T(abd)$, from which the parallelogram can be constructed. Some elementary properties of parallel segments are given in the following theorem.

THEOREM 16.

(i) If $ab \parallel cd$ then the points a, b, c and d are all distinct.
(ii) If $ab \parallel cd$ then $cd \parallel ab$.
(iii) If $ab \parallel pq$, $ab \parallel rs$ and $T(pqr)$ then $pq \parallel rs$.
(iv) If $ab \parallel cd$ then $ba \parallel cd$.
(v) If $ab \parallel cd$ then $ab \parallel dc$.
(vi) If $P(abcd)$ then $ab \parallel cd$ and $bc \parallel ad$.

To be well-behaved geometrically, we also need some form of Pappus' law as an axiom, which will, in turn, imply a form of Desargues' proposition.

Axiom (P). If $L(p_1p_2p_3)$, $L(q_1q_2q_3)$, $p_1q_2 \parallel p_2q_1$, and $p_2q_3 \parallel p_3q_2$ then $p_1q_3 \parallel p_3q_1$. *(Pappus' law)*

See Figure 7 for a standard case of Pappus' law.

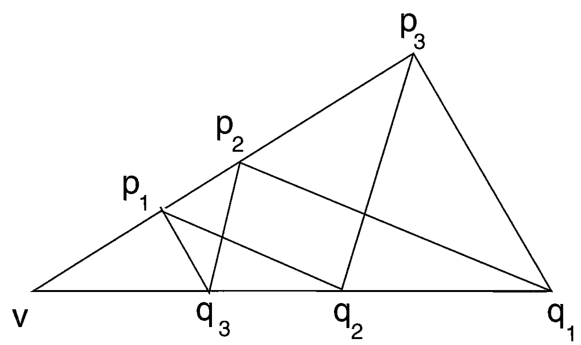

Figure 7.

The trapezoid construction $Z_v(abc) = d$ satisfies the following three axioms. Note that, unlike bisection and doubling, this construction is a partial function, defined only when the vertex v is distinct from the three points a, b and c which form a triangle and are thus noncollinear. The construction of the point $Z_v(abc) = d$ is shown in Figure 8.

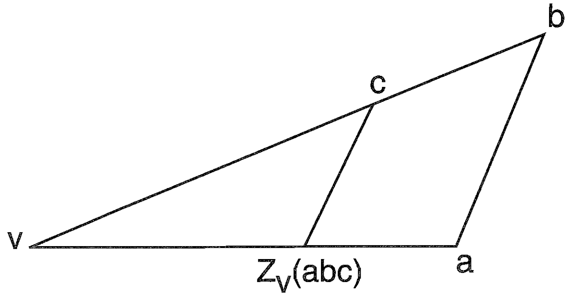

Figure 8.

(Z1) If $v \neq a, b, c$, $T(abc)$, $v|c|b$, $a|d|c$ and $ab \parallel cd$ then $Z_v(abc) = d$.

(Z2) If $v \neq a, b, c$, $T(abc)$ and $v|c|b$ then $v|Z_v(abc)|a$.

(Z3) If $v \neq a, b, c$, and $T(abc)$ then $ab \parallel cZ_v(abc)$.

We can define in the expected way the trapezoid relation in terms of the operation.

DEFINITION 4. If $v \neq a, b, c$ and $T(abc)$ then $Z_v(abcd)$ iff $Z_v(abc) = d$.

From this definition of the trapezoid relation and Axioms (Z1) and (Z2) the following elementary theorems follow at once.

THEOREM 17. If $Z_v(abcd)$ then $v|c|b$ and $v|d|a$.

THEOREM 18. If $Z_v(abcd)$ then $ab \parallel cd$.
THEOREM 19. If $v \neq a, b, c$, $T(abc)$, $v|c|b$, $v|d|a$ and $ab \parallel cd$ then $T_v(abcd)$.

Closely related to these theorems is the axiom of Pasch, which as stated by Hilbert for Euclidean geometry has the following content. If $T(abc)$, i.e., a, b and c form a triangle, and if a line L intersects the side ab, strictly between a and b, then L must intersect at least one of the other sides of the triangle. I strengthen one of Szmielew's Pasch forms, as Axiom (PC), to include a congruence result for trapezoids.

(PC) If $T(aa'b)$, $T(aa'b')$, $L(a'b'c')$, $a|b|c$ and $aa' \parallel bb' \parallel cc'$ then

(i) $a'|b'|c'$, (Pasch),

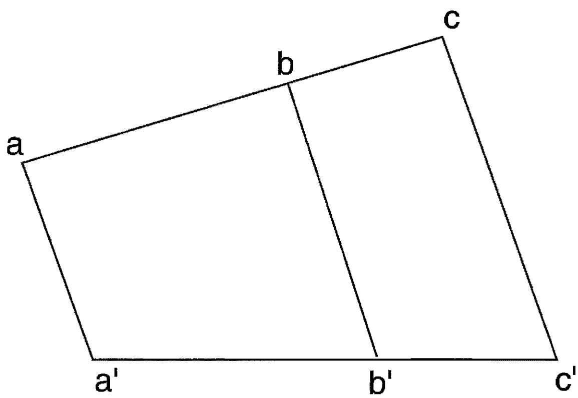

Figure 9.

(ii) If $ab \approx bc$ *then* $a'b' \approx b'c'$.

See Figure 9 to illustrate the axiom.

From Axiom (PC) we can derive the following special form (see Figure 10), which we need in the next section.

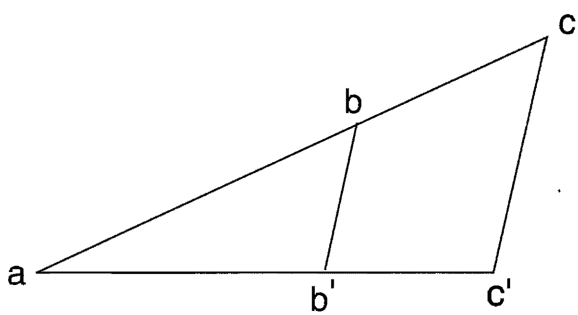

Figure 10.

THEOREM 20. If $T(abb')$, $L(ab'c')$, $a|b|c$ and $bb' \parallel cc'$ then

(i) $a|b'|c'$,

(ii) If $ab \approx bc$ then $ab' \approx b'c'$.

5. GEOMETRIC FOUNDATIONS OF RATIONAL NUMBERS

We can also use the constructive affine geometry developed here as a geometric foundation for constructing the rational numbers. From a conceptual standpoint we can think of carrying out a particular finite geometric construction for each given computation on fractions. (In this development I go no further than fractions; defining rational numbers as equivalence classes of fractions can then follow by standard methods, and in the same vein I consider here only nonnegative fractions.)

The standard approach to these matters is to use the points on a given line to be the numbers of a field and then use the original geometric constructions to define the field operations and the ordering relation. Using the field a coordinate system is defined, on the basis of which a Cartesian structure is constructed that satisfies the axioms. The heart of the representation theorem is then the assertion that any structure satisfying the axioms is isomorphic – in the present case with respect to the operations of bisection, doubling and constructing trapezoids –, to a Cartesian structure satisfying the axioms.

Here a still more constructivistic approach is adopted of considering only finite constructive configurations. Thus the representation of a positive rational number r, given as the fraction $\frac{p}{q}$, where p and q are positive integers, is a standard finite construction, which can be made clear by considering the example of the fraction $\frac{2}{3}$.

First, we are given as usual three noncollinear points with affine coordinates assigned as previously. We trisect the interval $\alpha\beta$ by doubling $\alpha\gamma$ to point $\alpha * \gamma$ and then doubling to point $\gamma * (\alpha * \gamma)$ The interval $\alpha(\gamma * (\alpha * \gamma))$ is divided into three affine congruent segments, easily proved using Axioms (BD) and (C5):

By (C5),

$$(1) \qquad \alpha(\alpha \oplus (\alpha * \gamma)) \approx (\alpha \oplus (\alpha * \gamma))(\alpha * \gamma),$$

but by (BD)

$$(2) \qquad \alpha \oplus (\alpha * \gamma) = \gamma,$$

and so,

$$(3) \qquad \alpha\gamma \approx \gamma(\alpha * \gamma),$$

as desired. A similar argument proves

$$(4) \qquad \gamma(\alpha * \gamma) \approx (\alpha * \gamma)(\gamma * (\alpha * \gamma)).$$

By the obvious trapezoid construction, as shown in Figure 11, we then construct $\delta(\frac{2}{3}, 0)$ by counting off three congruent segments between $\alpha = 0$ and $\beta = 1$, where $\delta = Z_\alpha((\alpha * \gamma)(\gamma * (\alpha * \gamma))\beta)$. The proof of congruence is the following.

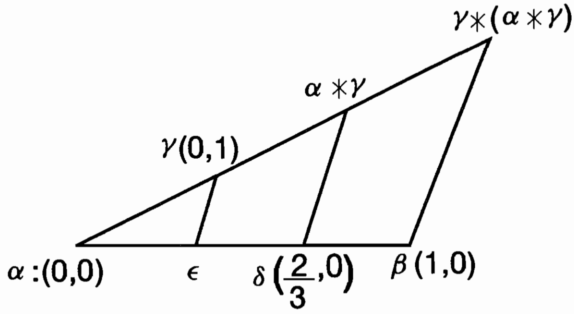

Figure 11.

By Axiom (PC), using especially the congruence part and Equation (4) above,

(5) $\epsilon\delta \approx \delta\beta,$

where $\epsilon = Z_\alpha(\gamma(\alpha * \gamma)\delta)$.

Similarly by use of Theorem 20, derived from Pasch's axiom, we infer

(6) $\alpha\epsilon \approx \epsilon\delta.$

Given this kind of construction for $\frac{p}{q}$, we use it to add and multiply positive rational numbers by applying the obvious equations

$$\frac{p}{q} + \frac{r}{s} = \frac{ps + qr}{qs}$$

and

$$\frac{p}{q} \times \frac{r}{s} = \frac{pr}{qs}.$$

The ordering relation for positive rational numbers is easily defined and is left to the reader.

APPENDIX: SUMMARY OF AXIOMS

(L1) It is not the case $L(\alpha\beta\gamma)$.

(L2) If $a = b$, $a = c$ or $b = c$, then $L(abc)$.

(L3) If $a \neq b$, $L(abp)$, $L(abq)$ and $L(abr)$ then $L(pqr)$.

(B1) $a \oplus a = a$ (Idempotency).

(B2) $a \oplus b = b \oplus a$ (Commutativity).

(B3) $(a \oplus b) \oplus (c \oplus d) = (a \oplus c) \oplus (b \oplus d)$ (Bicommutativity).

(B4) If $a \oplus b = a \oplus b'$ then $b = b'$ (Cancellation).

(D1) If $a * b = b * a$ then $a = b$ (Antisymmetry).

(D2) If $a * b = a * b'$ then $b = b'$ (Left Cancellation).

(D3) If $a * b = a' * b$ then $a = a'$ (Right Cancellation).

(BD) $a \oplus (a * b) = b$ (Reduction).

(LB) $L(ab(a \oplus b))$ (Linearity of Bisection).

(LL) If $L((a \oplus b)(b \oplus c)(a \oplus c))$ then $L(abc)$.

(Bt1) If $a|b|a$ then $a = b$ (Reflexivity).

(Bt2) If $a|b|c$ then $c|b|a$ (Symmetry).

(Bt3) If $a|b|c$ and $b|d|c$ then $a|b|d$ (Transitivity).

(Bt4) If $L(abc)$ then $a|b|c$ or $b|c|a$ or $c|a|b$ (Connectivity).

(BB) $a|a \oplus b|b$.

(BL) If $a|b|c$ then $L(abc)$.

(C1) If $aa \approx bc$ then $b = c$.

(C2) $ab \approx ba$.

(C3) If $ab \approx cd$ and $ab \approx ef$ then $cd \approx ef$.

(C4) If $a|b|c$, $a'|b'|c'$, $ab \approx a'b'$ and $bc \approx b'c'$ then $ac \approx a'c'$ (Additivity).

(C5) $a(a \oplus b) \approx b(a \oplus b)$.

(C6) $a * b \approx b * a$.

(C7) If $P(abcd)$ then $ab \approx cd$ and $bc \approx ad$.

(P) If $L(p_1p_2p_3)$, $L(q_1q_2q_3)$, $p_1q_2 \parallel p_2q_1$, and $p_2q_3 \parallel p_3q_2$ then $p_1q_3 \parallel p_3q_1$ (Pappus).

(Z1) If $v \neq a, b, c$, $T(abc)$, $v|c|b$, $a|d|c$ and $ab \parallel cd$ then $Z_v(abc) = d$.

(Z2) If $v \neq a, b, c$, $T(abc)$ and $v|c|b$ then $v|Z_v(abc)|a$.

(Z3) If $v \neq a, b, c$, and $T(abc)$ then $ab \parallel cZ_v(abc)$.

(PC) If $T(aa'b)$, $T(aa'b')$, $L(a'b'c')$, $a|b|c$ and $aa' \parallel bb' \parallel cc'$ then

(i) $a'|b'|c'$ (Pasch).

(ii) If $ab \approx bc$ then $a'b' \approx b'c'$.

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