

Quantifier-free axioms for constructive plane geometry ¹

Dedicated to A. Heyting on the occasion of his 70th birthday

by

Nancy Moler and Patrick Suppes

The purpose of this paper is to state a set of axioms for plane geometry which do not use any quantifiers, but only constructive operations. The relevant definitions and general theorems are stated; for reasons of space the proofs are only sketched. Quantifier-free arithmetic and, to some extent also, quantifier-free algebra have been the subjects of several investigations, but as far as we know, no prior set of quantifier-free axioms for plane geometry has been published. In a way, this omission is surprising, for an emphasis on geometric constructions has existed for a long time. The step of explicitly stating axioms in terms of the familiar constructions seems not to have been taken. In view of the highly constructive character of Euclidean geometry, it seems natural to strive for a formulation that eliminates all dependence on purely existential axioms, but not, of course, by the use of some wholly logical, non-geometric method of quantifier-elimination.

After we had already begun the work reported here, we found a very relevant and useful discussion of the constructions we take as primitive in the last chapter of Hilbert's well-known *Foundations of Geometry*, but Hilbert does not investigate their axiomatization. The axiomatic approach we have adopted here has been much influenced by Tarski (1959) and by Royden (1959). However, both Tarski and Royden consider only primitive concepts

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It is a pleasure to dedicate this paper to Professor Heyting on the occasion of his seventieth birthday. In view of his long interest in constructive mathematics and in geometry, we believe the subject of our paper makes it particularly appropriate to dedicate it to him.

that are relations. The use of constructions or operations as a primitive basis introduces certain complexities that are closely akin to division by zero in algebra. These complexities arise when the constructions are not well-defined in an intuitive sense. The variety of these conditions has probably been the greatest obstacle to rigorous axiomatization of geometric constructions.

The two primitive constructions in our theory are those of finding the intersection of two lines and of laying off or transferring segments. Because the representation theorem for models of our theory is in terms of vector spaces over Pythagorean fields, we refer to our theory as Pythagorean Constructive Geometry (PCG).

The constructive operation $S(xyuv) = w$, corresponding to the laying off of segments, is read: the point w is as distant from u in the direction of v as y is from x . The constructive operation $I(xyuv) = w$, corresponding in a limited sense with the drawing of straight lines, is read: the point w is collinear with the two points x and y and also collinear with the two points u and v ; in other words, w is the point of intersection of lines xy and uv . To simplify the presentation of the axioms, we use four defined relations which include one of Tarski's primitives, betweenness. Tarski's other primitive, equidistance, can be defined as $E(xyuv) \Leftrightarrow S(xyuv) = v$. To make the definitions and axioms easy to read at a glance, we use standard logical notation for the sentential connectives: \Leftrightarrow for *if and only if*, \rightarrow for *if \dots then*, \wedge for *and*, \vee for *or*, and \neg for *not*. Naturally, no quantifiers are introduced in the definiens of the four definitions, for otherwise our quantifier-free claim would be refuted by a statement of the axioms just in terms of the two primitive symbols.

DEFINITION 1. *Betweenness*

$$B(xyz) \Leftrightarrow [x \neq z \rightarrow S(xyaz) = y = S(zyax)] \wedge [x = z \rightarrow x = y]$$

DEFINITION 2. *Collinearity*

$$L(xyz) \Leftrightarrow S(xyaz) = y \vee S(zyax) = y \vee x = z$$

DEFINITION 3. *Noncollinearity of four points*

$$NL(xyuv) \Leftrightarrow \neg L(xyu) \wedge \neg L(yuv) \wedge \neg L(xuv) \wedge \neg L(xyv)$$

DEFINITION 4. *Distinctness*

$$NE(xyu) \Leftrightarrow x \neq y \wedge x \neq u \wedge u \neq y$$

Note that Definition 4 can obviously be extended to more variables.

In addition to the two operation symbols S and I , our axiomatization also depends on three individual constants α , β and γ , which stand for three points. Some simple axioms on these points are required to avoid one-point or three-point models of our axioms.

By a *model of Pythagorean Constructive Geometry* we understand a structure $\mathfrak{A} = \langle A, S, I, \alpha, \beta, \gamma \rangle$ such that (i) A is a set containing at least the three points α , β and γ , (ii) S and I are four-place operations on A , and (iii) all of the following eighteen axioms are satisfied by any points t, u, v, w, x, y and z in A .

AXIOM 1. *Lower-dimension Axiom*

$$S(\beta\alpha\beta\gamma) \neq \alpha \wedge S(\gamma\beta\gamma\alpha) \neq \beta \wedge S(\alpha\gamma\alpha\beta) \neq \gamma$$

AXIOM 2. *Nondegeneracy Axiom*

$$S(\alpha\beta\beta\gamma) \neq \gamma$$

AXIOM 3. *Reflexivity Axiom for Segment Construction*

$$S(xy\gamma\alpha) = x$$

AXIOM 4. *Identity Axiom for Segment Construction*

$$S(x\alpha w) = w$$

AXIOM 5. *Transitivity Axiom for Segment Construction*

$$S(xywt) = z \rightarrow S(xyuv) = S(zwuv)$$

AXIOM 6. *Direction Axiom*

$$S(xyuv) = z \rightarrow S(vzvu) = z \vee S(zvzu) = v$$

AXIOM 7. *Distance Axiom*

$$B(uvw) \wedge S(uvwu) = z \rightarrow S(zuv) = u$$

AXIOM 8. *Connectivity Axiom*

$$(u \neq v) \wedge S(uvw) = v \wedge S(uvw) = v \rightarrow S(uvw) = z$$

AXIOM 9. *First Transitivity Axiom for Betweenness*

$$B(uvz) \wedge B(vwz) \rightarrow B(uvw)$$

AXIOM 10. *Second Transitivity Axiom for Betweenness*

$$(v \neq w) \wedge B(uvw) \wedge B(vwz) \rightarrow B(uvz)$$

AXIOM 11. *Five-segment Axiom*

$$(u \neq v) \wedge B(xy\alpha) \wedge B(uvw) \wedge S(uvxy) = y = S(vwxy) \\ \wedge S(ut\alpha) = s = S(vty) \rightarrow S(vt\alpha) = s$$

AXIOM 12. *First Commutative Axiom for Line Intersection*

$$I(xyuv) = I(uvxy)$$

AXIOM 13. *Second Commutative Axiom for Line Intersection*

$$I(xyuv) = I(xyvu)$$

AXIOM 14. *Collinearity Axiom*

$$(x \neq y) \wedge I(xyuv) = w \wedge [S(xyaw) = y \vee S(ywya) = w] \\ \rightarrow S(uvuw) = v \vee S(vwvu) = w$$

AXIOM 15. *Pasch's Axiom*

$$(t \neq z) \wedge B(x, t, I(xtyz)) \wedge B(y, I(xtyz), z) \rightarrow B(x, I(xyzt), y)$$

AXIOM 16. *Axiom for Regular Line Intersection*

$$(u \neq v) \wedge L(xwy) \wedge L(uvw) \wedge [-L(xyu) \vee -L(xyv)] \rightarrow I(xyuv) = w$$

AXIOM 17. *Euclid's Axiom*

$$NL(xyuv) \wedge B(x, I(x, S(xyuv), y, u), S(xyuv)) \\ \wedge S(y, S(xyuv), x, u) \neq u \rightarrow L(x, y, I(xyuv))$$

AXIOM 18. *Upper-dimension Axiom*

$$NE(xyuv) \rightarrow B(u, I(uvxy), v) \vee B(u, I(uaxy), x) \vee B(v, I(vxuy), x)$$

Axiom 1 insures that α , β , γ are three distinct, noncollinear points and thus that the dimension of the space is greater than or equal to two. If the distances between these three points were equal, it would be impossible to construct any new points using the operations S and I . Thus, Axiom 2 states that the distances $\alpha\beta$ and $\beta\gamma$ are unequal.

Axioms 3, 4 and 5 express obvious conditions. Axioms 6 and 8 combine to formalize the concept that $S(xyuv)$ is in fact on the same side of u as v . Together they prove Tarski's (1959) axiom of connectivity. Axiom 7 states that a segment is longer than any of its parts: that in fact the segment vz is congruent to the segment uv . Axiom 7 is also important in extending the line segment so that for all x and y there is a w such that $B(xyw)$. Both transitivity axioms for Betweenness (Axioms 9 and 10) appear to be essential for this extension. Once this extension has been made, one uses the strong form of Pasch's Axiom (Axiom 15), discussed below, to prove that for all x and y there is a w such that $B(xwy)$. We feel that there may be a simpler way of stating Axioms 9 and 10 so that only the primitive symbol S is used.

Axiom 11 is a direct translation from the similar axiom (A9) of Tarski (1959). This axiom is important in proving addition and subtraction of segments. Axiom 18, which we had developed before finding a similar axiom in Szczerba and Tarski (1964) and Axiom 11 combine to prove theorems relating to the congruence of triangles, particularly the side-side-side theorem.

Axiom 14 shows that if the point of intersection of the two lines xy and uv is collinear with one line it must be collinear with the other. Axiom 15 is the weak form of Pasch's Axiom as stated by Veblen (1904). With axiom 14 it can be shown that $B(z, t, I(xyzt))$ also holds and thus the stronger form of A7 in Tarski (1959) can be derived. Axiom 16 states that if xy and uv are two distinct lines with one point in common, then that point is the point of intersection.

It has become common to use Playfair's Axiom as the desirable equivalent to Euclid's Axiom. However, when dealing with constructions rather than relations as primitives, it is desirable to have some criteria for deciding that two lines actually intersect and where that point of intersection is. The form of Euclid's Axiom we have used, Axiom 17, is that if the distance between two lines increases or decreases, the two lines must intersect. The axiom appears complicated in form but gives an easily applied criterion for intersection. With the Upper-dimension Axiom and Pasch's Axiom one can determine the location of the point of intersection with respect to the other two points given on the line. Playfair's Axiom can then be derived with the help of the other axioms.

We have not investigated the independence of the axioms, but we feel that some simplification can be made.

A brief look at the axioms will show that for some x, y, u , and v , $I(xyuv)$ is not actually determined. For example, $I(xxyu)$ is not determined. Similarly, the construction $S(xyuu)$ is not determined if $x \neq y$. To make our viewpoint completely explicit, we assume that the operations S and I are *defined* for all quadruples of points in A , but undecidable or undetermined by the theory in the degenerate or nonstandard cases. For instance, $S(xyuu)$ is some point in A , since S is always defined, but what point it is cannot be determined from the axioms. To handle these conditions in the representation theorem, we extend the concept of isomorphism to include undecidable cases. In the definition that follows, it is required that A be a nonempty set and 0 a quaternary operation on A in order for $\mathfrak{A} = \langle A, 0 \rangle$ to be a model.

Let $\mathfrak{A} = \langle A, 0 \rangle$ be a model of a theory T . The operation 0 is *undecidable* with respect to condition $\varphi(xyuw)$ if and only if the sentence

$$\varphi(xyuw) \rightarrow 0(xyuw) = z$$

is undecidable in T . Let $\mathfrak{A} = \langle A, 0 \rangle$ and $\mathfrak{A}' = \langle A', 0' \rangle$ be two models of a theory T . Then \mathfrak{a} and \mathfrak{a}' are *isomorphic* (with respect to T) if and only if there is a function f such that:

1. The domain of f is A and the range of f is A' ;
2. f is a one-one function;
3. For all x, y, u and v in A such that the operation 0 is not undecidable with respect to any condition $\varphi(xyuw)$, we have

$$f(0(xyuw)) = 0'(f(x)f(y)f(u)f(v));$$

4. Given any condition $\varphi(xyuw)$ with respect to which 0 is undecidable, there is a unique condition φ' with respect to which $0'$ is undecidable, and conversely, such that

$$\varphi(xyuw) \leftrightarrow \varphi'(f(x)f(y)f(u)f(v)).$$

Using these definitions we may easily prove the following lemma.

LEMMA 1: *Let $\mathfrak{A} = \langle A, S, I, \alpha, \beta, \gamma \rangle$ be a model of PCG. Then the operation S is undecidable with respect to the condition $\varphi(xyuw)$, where*

$$\varphi(xyuw) \Leftrightarrow x \neq y \wedge u = v.$$

The operation I is undecidable with respect to the condition $\Gamma_1 \vee \Gamma_2 \vee \Gamma_3$ where

$$\Gamma_1(xyuw) \Leftrightarrow x = y \vee u = v$$

$$\Gamma_2(xyuw) \Leftrightarrow L(xyu) \wedge L(xyv)$$

$$\Gamma_3(xyuw) \Leftrightarrow NL(xyuw) \wedge B(x, I(x, S(xyuw), y, u), S(xyuw)) \\ \wedge S(y, S(xyuw), x, u) = u.$$

Γ_3 states the conditions under which xy is parallel to uw .

Our Representation Theorem is stated in terms of a special Cartesian space over an ordered Pythagorean field, which we now define. Let $\mathfrak{F} = \langle F, +, \cdot, \leq \rangle$ be an ordered field. Then \mathfrak{F} is a Pythagorean field if for all a and b in F , $\sqrt{a^2 + b^2}$ is in F . Now let $A_{\mathfrak{F}} = F \times F$. We define the operations $S_{\mathfrak{F}}$ and $I_{\mathfrak{F}}$, for all x, y, u, v , and w in $A_{\mathfrak{F}}$, in the following manner:

$$S_{\mathfrak{F}}(xyuw) = w \Leftrightarrow w = \langle w_1, w_2 \rangle,$$

where

$$w_1 = \begin{cases} u_1 & \text{if } x = y, \\ u_1 - (u_1 - v_1) \left(\frac{(x_1 - y_1)^2 + (x_2 - y_2)^2}{(u_1 - v_1)^2 + (u_2 - v_2)^2} \right)^{\frac{1}{2}} & \text{if } x \neq y, \end{cases}$$

$$w_2 = \begin{cases} u_2 & \text{if } x = y, \\ u_2 - (u_2 - v_2) \left(\frac{(x_1 - y_1)^2 + (x_2 - y_2)^2}{(u_1 - v_1)^2 + (u_2 - v_2)^2} \right)^{\frac{1}{2}} & \text{if } x \neq y, \end{cases}$$

$$I_{\mathfrak{F}}(xyuv) = w \Leftrightarrow w = \langle w_1, w_2 \rangle$$

where

$$w_1 = \frac{(u_1 - v_1)(y_1 x_2 - y_2 x_1) - (x_1 - y_1)(v_1 u_2 - v_2 u_1)}{(u_1 - v_1)(x_2 - y_2) - (u_2 - v_2)(x_1 - y_1)}$$

and w_2 is defined with the obvious symmetry.

We choose $\alpha_{\mathfrak{F}}$, $\beta_{\mathfrak{F}}$ and $\gamma_{\mathfrak{F}}$ as three distinct points of $A_{\mathfrak{F}}$ such that

$$S_{\mathfrak{F}}(\alpha_{\mathfrak{F}}\beta_{\mathfrak{F}}\beta_{\mathfrak{F}}\gamma_{\mathfrak{F}}) \neq \gamma_{\mathfrak{F}}$$

and

$$S_{\mathfrak{F}}(\beta_{\mathfrak{F}}\alpha_{\mathfrak{F}}\beta_{\mathfrak{F}}\gamma_{\mathfrak{F}}) \neq \alpha_{\mathfrak{F}} \wedge S(\gamma_{\mathfrak{F}}\beta_{\mathfrak{F}}\gamma_{\mathfrak{F}}\alpha_{\mathfrak{F}}) \neq \beta_{\mathfrak{F}} \wedge S(\alpha_{\mathfrak{F}}\gamma_{\mathfrak{F}}\alpha_{\mathfrak{F}}\beta_{\mathfrak{F}}) \neq \gamma_{\mathfrak{F}}.$$

From the definition of $S_{\mathfrak{F}}$ it can easily be shown that such points exist, for example, let $\alpha_{\mathfrak{F}} = \langle 0, 0 \rangle$, $\beta_{\mathfrak{F}} = \langle 0, 1 \rangle$ and $\gamma_{\mathfrak{F}} = \langle 1, 0 \rangle$. The structure $\mathfrak{A}_{\mathfrak{F}} = \langle A_{\mathfrak{F}}, S_{\mathfrak{F}}, I_{\mathfrak{F}}, \alpha_{\mathfrak{F}}, \beta_{\mathfrak{F}}, \gamma_{\mathfrak{F}} \rangle$ is called the Constructive Cartesian Space over the ordered Pythagorean field \mathfrak{F} , and it is easily shown that it is a model of PCG.

LEMMA 2. *Let $\mathfrak{A}_{\mathfrak{F}}$ be the Constructive Cartesian Space over the ordered Pythagorean field \mathfrak{F} . Then the operation $S_{\mathfrak{F}}$ is undecidable with respect to the condition $\varphi_{\mathfrak{F}}(xyuv)$ where*

$$\varphi_{\mathfrak{F}}(xyuv) \Leftrightarrow x \neq y \wedge u = v.$$

The operation $I_{\mathfrak{F}}$ is undecidable with respect to the condition $\Gamma_{\mathfrak{F}}(xyuv)$ where

$$\Gamma_{\mathfrak{F}}(xyuv) \Leftrightarrow (u_1 - v_1)(x_2 - y_2) = (u_2 - v_2)(x_1 - y_1).$$

We are now ready to state and sketch the proof of the Representation Theorem.

Representation Theorem. *For a structure $\mathfrak{A} = \langle A, S, I, \alpha, \beta, \gamma \rangle$ to be a model of PCG it is necessary and sufficient that \mathfrak{A} be isomorphic to the Constructive Cartesian Space $\mathfrak{A}_{\mathfrak{F}}$ over some ordered Pythagorean field \mathfrak{F} .*

PROOF. As already remarked, it is easy to verify that all of the axioms of PCG hold in $\mathfrak{A}_{\mathfrak{F}}$ and thus $\mathfrak{A}_{\mathfrak{F}}$ is a model of PCG. Thus if \mathfrak{A} is isomorphic to $\mathfrak{A}_{\mathfrak{F}}$ then \mathfrak{A} must also be a model.

Now assume \mathfrak{A} is a model of PCG. We first must construct an ordered field \mathfrak{F} . The elements of this field are taken to be the points on the line through α and β , that is, $F = \{x \in A : L(\alpha\beta x)\}$. The definitions of operations in the field will involve several geometric constructions.

The construction $E(xy) = z$ extends the line segment xy through x to z so that x is the midpoint of yz . From Axiom 2 we know that $S(\alpha\beta\beta\gamma) \neq \gamma$. The axioms are so stated that either $B(\alpha\beta S(\beta\gamma\alpha\beta))$ or $B(\beta\gamma S(\alpha\beta\beta\gamma))$. For definiteness, but without loss of generality, let us assume the former.

$$\begin{aligned} E(xy) = z \Leftrightarrow \{ & [x = y \rightarrow x = z] \\ & \wedge [S(\alpha\beta yx) = x \rightarrow S(y, x, x, S(\beta\gamma yx)) = z] \\ & \wedge [y \neq x \wedge S(\alpha\beta yx) \neq x \\ & \wedge B(y, x, S(\alpha\beta yx)) \rightarrow S(y, x, x, S(\alpha\beta yx)) = z] \\ & \wedge [S(\alpha\beta yx) \neq x \wedge B(y, S(\alpha\beta yx), x) \\ & \rightarrow S(y, x, x, S(y, x, S(\alpha\beta yx), x)) = z]\}. \end{aligned}$$

If x , y and z are not collinear, a point w so that xy and zw are parallel is given by

$$P(xyz) = w \Leftrightarrow E(y, E(xz)) = w.$$

If x , y and z are collinear then w is a point on the line.

A third construction, $R(xy) = z$, can be defined so that the angle xyz is a right angle. We omit the definition, but note that if $x = y$ then $x = z$. (The given points α , β and γ are used to make the point z unique.)

We also define the following generalization of betweenness to indicate direction.

$$D(xyz) \Leftrightarrow B(xyz) \vee B(xzy).$$

Using these constructions, the relation \leq and the operation $+$ are defined for the elements of F by

$$\begin{aligned} x+y = v \Leftrightarrow \{ & [B(\alpha xy) \rightarrow S(\alpha xy E(y\alpha)) = v] \\ & \wedge [B(\alpha yx) \rightarrow S(\alpha yx E(x\alpha)) = v] \\ & \wedge [\neg D(\alpha xy) \rightarrow S(\alpha xy\alpha) = v]\} \\ x \leq y \Leftrightarrow \{ & x = y \vee [B(x\alpha\beta) \wedge S(yx\alpha\beta) = y] \\ & \vee [B(\alpha xy) \wedge S(\alpha x\alpha\beta) = x \wedge S(\alpha y\alpha\beta) = y]\}. \end{aligned}$$

In order to define the operation \cdot it is necessary to use points outside of F in the definiens. One such point is $R(\beta\alpha) = \xi$.

$$x \cdot y = v \Leftrightarrow [D(\alpha xy) \rightarrow S[\alpha, I(x, P(\beta S(\alpha y \alpha \xi)x), \xi, \alpha), \alpha, \beta] = v] \\ \wedge [\neg D(\alpha xy) \rightarrow S[\alpha, I(xP(\beta S(\alpha y \alpha \xi)x), \xi, \alpha), \alpha, E(\alpha\beta)] = v].$$

Further constructions are used in proving that $\mathfrak{F} = \langle F, +, \cdot, \leq \rangle$ is an ordered Pythagorean field, but these constructions will not be defined here.

In order to show that \mathfrak{A} is isomorphic to $\mathfrak{A}_{\mathfrak{F}}$ a rectangular coordinate system is defined for all x in A by

$$x_1 = I(x, P(\xi, \alpha, x), \alpha, \beta) \\ x_2 = I(x, P(\beta, \alpha, x), \alpha, \xi)$$

This coordinate system is used to define a one-one correspondence f between the elements x in A and the coordinates, $\langle x_1, x_2 \rangle$ in $F \times F$.

The proof is completed by showing that when no undecidable condition holds for points x, y, u and v in A then

$$f(S(xyuv)) = S_{\mathfrak{F}}(f(x)f(y)f(u)f(v)),$$

and similarly for $I(xyuv)$. Secondly, equivalence of undecidable conditions for S and $S_{\mathfrak{F}}$, and for I and $I_{\mathfrak{F}}$ is easily shown from the lemmas already stated.

Finally, we turn to the uniqueness theorem whose proof follows standard lines and is therefore omitted. One preliminary definition is needed. A transformation $T(x)$ is a *generalized Euclidean* transformation of $F \times F$ onto itself if $T(x) = sPx + Q$ where s is in F , P is an orthogonal 2×2 matrix of elements of F and Q is in $F \times F$.

Uniqueness Theorem. Let \mathfrak{A} be a model of PCG isomorphic under mappings f and g to the Constructive Cartesian Space $\mathfrak{A}_{\mathfrak{F}}$ for some ordered Pythagorean field \mathfrak{F} . Then there is a generalized Euclidean transformation T of $F \times F$ onto itself such that $g = T \circ f$, where \circ is function composition.

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Stanford University
Stanford, California