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SOME ASYMPTOTIC PROPERTIES OF LUCE'S BETA LEARNING MODEL*

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This paper studies asymptotic properties of Luce's beta model. Asymptotic results are given for the two-operator and four-operator cases of contingent and noncontingent reinforcement.

For application to various simple learning situations, Luce and his collaborators, Bush and Galanter, [1, 7] have considered a learning model in which the changes in probability of response from trial to trial are not linear functions of the probability of response on the preceding trial. Both theoretical and empirical considerations have motivated the development of the beta model. Some learning theorists like Hull and Spence believe that overt response behavior may best be explained in terms of a construct like that of response strength. From this viewpoint stochastic learning models which postulate a linear transformation of the probability of response from one trial to the next, with the transformation depending on the reinforcing event, are unsatisfactory in so far as they offer no more general psychological justification of their postulates. From an empirical standpoint there is evidence in some experiments, particularly certain T-maze experiments with rats, that the linear stochastic models do not yield good predictions of actual behavior [1, 7].

On the basis of some very simple postulates [7] on choice behavior, Luce has shown that there exists a ratio scale v over the set of responses with the property that

$$p_{i,n} = \frac{v_n(i)}{\sum_i v_n(i)} ,$$

where $p_{i,n}$ is the probability of response A_i on trial n, and $v_n(i)$ is the strength of this response on trial n. Additional simple postulates lead to the result that the $v_n(i)$ are transformed linearly from trial to trial, and this unobservable stochastic process on response strengths then determines a stochastic process

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in the response probabilities. Superficially, it would seem that the simplest way to study the asymptotic behavior of the response probabilities—a subject of interest in connection with nearly any learning data—would be to determine the asymptotic behavior of the response strengths $v_n(i)$ and then infer by means of the equation given above the behavior of the response probabilities. This course is pursued rather far by Luce [7] and encounters numerous mathematical difficulties. We have taken the alternative path of studying directly the properties of the nonlinear transformations on the response probabilities to obtain results on their asymptotic behavior.

We restrict ourselves to situations in which one of two responses, A_1 and A_2, is made. Let p_n be the probability of response A_1 on trial n, and let E_1 be the event of reinforcing response A_1, and E_2 the event of reinforcing response A_2.

Luce's beta model is then characterized by the following transformations: if A_j and E_k occurred on trial n, then for $j = 1, 2$ and $k = 1, 2$,

$$p_{n+1} = \frac{p_n}{p_n + \beta_{jk}(1 - p_n)}, \tag{1}$$

where $\beta_{jk} > 0$. Luce [7] gives a more general formulation. (Generally, we want $\beta_{j1} < 1$ and $\beta_{j2} > 1$, to reflect the primary effects of reinforcement; moreover, it is ordinarily assumed that $\beta_{11} < \beta_{21} < \beta_{12} < \beta_{22}$.) Throughout this paper it is assumed that $0 \neq p_1 \neq 1$.

The most important fact about (1) is that the operators commute. For example, suppose in the first n trials there are b_1 occurrences of A_1E_1, b_2 occurrences of A_2E_1, b_3 occurrences of A_1E_2, b_4 occurrences of A_2E_2; then it is easily shown that

$$p_{n+1} = \frac{p_1}{p_1 + \beta_{11}^{b_1}\beta_{21}^{b_2}\beta_{12}^{b_3}\beta_{22}^{b_4}(1 - p_1)}. \tag{2}$$

The aim of the present paper is to study asymptotic properties of the beta model for certain standard probabilistic schedules of reinforcement. The methods of attack used by Karlin [4] and by Lamperti and Suppes [6] for linear learning models do not directly apply to the nonlinear beta model.

The basis of our approach is to change the state space (the probability p_n is the state) from the unit interval to the whole real line in such a way that the transformations (1) become simply translations. The noncontingent case (the next section) then reduces to sums of independent random variables; the contingent cases can also be studied by "comparing" the resulting random walks with the case of sums of random variables. The probabilistic tool for this is developed and applied in later sections. The general conclusion to be drawn from our results is that for all but one case of noncontingent reinforcement individual response probabilities are ultimately either zero or one, which is in marked contrast to corresponding results for linear learning models. Absorption at zero or one also occurs for many, but not all, cases of contingent reinforcement.

Noncontingent Reinforcement with Two Operators

If the probability of a reinforcement is independent of response and trial number, we have what is called simple noncontingent reinforcement. Let π be the probability of an E_1 reinforcement, and for simplicity let

$$(3) \qquad \begin{cases} \beta_{11} = \beta_{21} = \beta, \\ \beta_{12} = \beta_{22} = \gamma, \\ 0 < \beta < 1, \\ \gamma > 1. \end{cases}$$

We seek an expression for the asymptotic probability distribution of response probabilities in terms of the numbers π, β, and γ.

The random variable η_n is defined recursively as follows:

$$\eta_1 = \begin{cases} \beta & \text{with prob } \pi, \\ \gamma & \text{with prob } (1-\pi); \end{cases}$$

$$\eta_{n+1} = \begin{cases} \eta_n\beta & \text{with prob } \pi, \\ \eta_n\gamma & \text{with prob } (1-\pi). \end{cases}$$

The random variable X_n is defined as follows:

$$X_n = \log \eta_n .$$

Then

$$(4) \qquad X_{n+1} = \begin{cases} X_n + \log \beta & \text{with prob } \pi, \\ X_n + \log \gamma & \text{with prob } (1-\pi). \end{cases}$$

It is clear from (4) and what has preceded that X_n is the sum of n independent identically distributed random variables Y_i defined by

$$Y_i = \begin{cases} \log \beta & \text{with prob } \pi, \\ \log \gamma & \text{with prob } (1-\pi). \end{cases}$$

By the strong law of large numbers, with probability one as $n \to \infty$

$$(5) \qquad \begin{aligned} X_n &\to \infty \quad \text{if} \quad \pi \log \beta + (1-\pi) \log \gamma > 0, \\ X_n &\to -\infty \quad \text{if} \quad \pi \log \beta + (1-\pi) \log \gamma < 0. \end{aligned}$$

Define now for any real number x

$$(6) \qquad F_x(p_1) = \frac{p_1}{p_1 + e^x(1-p_1)}.$$

Then $p_{n+1} = F_{X_n}(p_1)$ for the sequence of reinforcements η_n, where $X_n = \log \eta_n$. These results are utilized to prove the following theorem.

THEOREM 1. *Let* $c = \pi \log \beta + (1 - \pi) \log \gamma$. *Then with probability one*

$$p_\infty = \begin{cases} 0 & \textit{if} \quad c > 0, \\ 1 & \textit{if} \quad c < 0. \end{cases}$$

If $c = 0$, *then* p_n *oscillates between* 0 *and* 1, *so that with probability one*

$$\limsup p_n = 1$$

$$\liminf p_n = 0.$$

Despite this oscillation, there is a limiting distribution for p_n ; *it is concentrated at* 0 *and* 1 *with equal probabilities* $\frac{1}{2}$.

PROOF. The results for $c > 0$ and $c < 0$ follow immediately from (5), (6), and the remark following. In case $c = 0$, note that $E(Y_i) = 0$. It is known [2] that the sums X_n are then recurrent; that is, they repeatedly take on values arbitrarily close to any possible value. In particular, X_n takes on repeatedly arbitrarily large and arbitrarily small values (with probability one), which upon recalling (6) proves the second statement. The third statement is a consequence of the central limit theorem, which implies that for any A, $\Pr(X_n > A)$ and $\Pr(X_n < -A)$ both converge to one-half as n increases. Again the assertion of the theorem follows from this fact and (6).

Two Theorems on Random Walks

The results of this section are special cases of those in [5]. However, the present approach has the advantages of simplicity and directness.

We have seen that the two-operator, noncontingent beta model gives rise to a Markov process on the real line such that from x the "moving particle" goes to $x + a$ or $x - b$ with (constant) probabilities φ and $1 - \varphi$. The contingent case leads to a similar process, except that the transition probabilities become functions of x. The four-operator model gives rise to a process with four possible transitions, from x to $x + a_i$, say, $i = 1, 2, 3, 4$. In this section some simple results on processes of these sorts will be obtained, in preparation for the study of the more general cases of the beta model. In the interest of clarity, only the two-operator case will be treated in full; the more general case can be handled in a similar way, but the details are cumbersome. Our approach was suggested by the work of Hodges and Rosenblatt [3].

Let $\{X_n\}$ be a real Markov process such that if $X_n = x$,

$$X_{n+1} = \begin{cases} x + a & \text{with prob } \varphi(x), \\ x - b & \text{with prob } [1 - \varphi(x)], \end{cases} \tag{9}$$

where $0 < a, b, \varphi(x), 1 - \varphi(x)$. Let $\{Y_n\}$ be another process of the same type (and with the same a and b) but with constants θ and $1 - \theta$ as the transition probabilities in place of $\varphi(x)$ and $1 - \varphi(x)$.

LEMMA. *If for all* $x \geq M$, *one has* $\varphi(x) \geq \theta$, *and if* $\Pr(Y_n \to +\infty) > 0$, *then* $\Pr(X_n \to +\infty) > 0$. *If, on the other hand, for* $x \geq M$, $\varphi(x) \leq \theta$ *and if* $\Pr(Y_n \to +\infty) = 0$, *then* $\Pr(X_n \to +\infty) = 0$.

PROOF. Let $\{\xi_n\}$ be a sequence of independent random variables, each uniformly distributed on $[0, 1]$. The $\{X_n\}$ process will be referred to $\{\xi_n\}$ by letting

$$X_{n+1} = \begin{cases} X_n + a & \text{if} \quad \xi_{n+1} \leq \varphi(X_n), \\ X_n - b & \text{otherwise.} \end{cases} \tag{10}$$

This does lead to the transition law (9) as may easily be seen. The $\{Y_n\}$ process can be linked to $\{X_n\}$ by referring it after the manner of (10) to the *same* sequence $\{\xi_n\}$, so that $Y_{n+1} = Y_n + a$ if and only if $\xi_{n+1} \leq \theta$.

Choose $Y_0 > M$. Whatever the value of X_0, since $\varphi(x) > 0$ there is positive probability that $X_m \geq Y_0$ for some m; therefore assume $X_0 \geq Y_0$. We now assert that for those sequences $\{Y_n\}$ with the property that $Y_n \geq M$ for all n, the inequality $X_n \geq Y_n$ is also valid for all n. This follows from our construction "linking" the processes, and the assumption that $\varphi(x) \geq \theta$ for $x \geq M$; the transition $X_{n+1} = X_n - b$ and $Y_{n+1} = Y_n + a$ is impossible, so $X_n - Y_n$ can only increase.

To complete the proof, note that since $\Pr(Y_n \to +\infty)$ is positive, so is $\Pr(Y_n \to +\infty, Y_n \geq M \text{ for all } n)$. But the event $Y_n \to +\infty$, $Y_n \geq M$ for all n may be considered as a set S in the sample space of the sequence $\{\xi_n\}$; S is a set of positive probability, and is contained in the set $X_n \to \infty$ since on S, $X_n \geq Y_n$ and $Y_n \to \infty$. Hence $\Pr(X_n \to +\infty) > 0$. The second part of the lemma is proved in a similar way, using the same construction linking $\{X_n\}$ and $\{Y_n\}$.

THEOREM 2. *Let* $b/(a + b) = c$, *and suppose that*

$$\lim_{x \to +\infty} \varphi(x) = \alpha \quad \textit{and} \quad \lim_{x \to -\infty} \varphi(x) = \beta \tag{11}$$

exist. Then if $\alpha < c$ *and* $\beta > c$,

$$\Pr(\limsup X_n = +\infty, \liminf X_n = -\infty) = 1 \quad (\{X_n\} \textit{ is recurrent}), \tag{12}$$

while if $\alpha < (>)\, c$ *and* $\beta < (>) c$, *then*

$$\Pr(X_n \to -\infty\ (+\infty)) = 1. \tag{13}$$

Finally, if $\alpha > c$ *and* $\beta < c$,

$$\Pr(X_n \to +\infty) = \delta, \qquad \Pr(X_n \to -\infty) = 1 - \delta \tag{14}$$

for some $0 < \delta < 1$.

PROOF. Suppose, for instance, that $\alpha < c$. Let $\{Y_n\}$ (as in the lemma) be a process with constant transition probabilities θ and $1 - \theta$ where $\alpha < \theta < c$. The $\{Y_n\}$ process may be regarded as sums of random variables

$$Y_n = Y_0 + \sum_{i=1}^{n} Z_i , \quad \text{where} \quad \Pr(Z_i = a) = \theta \quad \text{and} \qquad (15)$$
$$\Pr(Z_i = -b) = 1 - \theta.$$

But $E(Z_i) = a\theta - b(1 - \theta) < 0$, since $\theta < c$; this implies that $\Pr(Y_n \to -\infty) = 1$ by the law of large numbers. From the lemma, $\Pr(X_n \to +\infty) = 0$.

Similarly, if $\alpha > c$ it follows that $\Pr(X_n \to +\infty) > 0$. Since the lemma also holds for convergence to $-\infty$ (with φ and θ replaced by $1 - \varphi$ and $1 - \theta$), we obtain in the same way that $\beta < c$ makes $\Pr(X_n \to -\infty) > 0$, while if $\beta > c$ this probability is zero.

Consider the case when $\alpha < c$ and $\beta < c$; there is then positive probability of absorption at $-\infty$, but not at $+\infty$. It is not hard to see that $X_n \to -\infty$ with probability one; the idea is roughly as follows. Since $X_n \nrightarrow +\infty$, we have $X_n \leq N$ infinitely often with probability arbitrarily close to 1 for some N. Now the probability that from or to the left of N the random walk goes and remains to the left of $N - M$ must be positive since $\Pr(X_n \to -\infty) > 0$. But in an infinite sequence of not necessarily independent trials, an event whose probability on each trial is bounded away from zero is certain to occur. Hence for any M, the random walk will eventually become and remain to the left of $N - M$, and therefore $X_n \to -\infty$ with probability arbitrarily close to 1 (and so equal to one). The other cases are similar; one can think of $\alpha > c$ or $\alpha < c$ as the conditions under which $+\infty$ is an absorbing or reflecting barrier, etc., and the process behaves accordingly.

The generalization to the four-operator case will now be described. Let $\{X_n\}$ be a real Markov process such that if $X_n = x$, then

$$X_{n+1} = x + a_i \quad \text{with prob } \varphi_i(x), \qquad (17)$$

where a_1, $a_2 > 0 > a_3$, a_4 and $\varphi_i(x) > 0$. Suppose

$$\lim_{x \to +\infty} \varphi_i(x) = \alpha_i \quad \text{and} \quad \lim_{x \to -\infty} \varphi_i(x) = \beta_i \qquad (18)$$

exist, and let

$$\mu_+ = \sum_{i=1}^{4} a_i \alpha_i \quad \text{and} \quad \mu_- = \sum_{i=1}^{4} a_i \beta_i .$$

By methods entirely similar to those used above, but rather more involved, it is possible to prove the following.

THEOREM 3. *For the process $\{X_n\}$ described above, if $\mu_+ < 0$ and $\mu_- > 0$ then* (12) *holds; if $\mu_+ < (>) 0$ and $\mu_- < (>) 0$ then* (13) *applies; while if $\mu_+ > 0$ and $\mu_- < 0$,* (14) *is valid.*

Contingent Reinforcement with Two Operators

If the probability of reinforcement depends only on the immediately preceding response (on the same trial), one has (*simple*) *contingent reinforcement.* Let $\Pr(E_1 \mid A_1) = \pi_1$ and $\Pr(E_1 \mid A_2) = \pi_2$, and let the two operators β and γ be specified as in (3). Using (6), define the random variable X_n recursively. (Note that $\log \gamma$ appears first, since $\log \gamma > 0$ and $\log \beta < 0$, in order most directly to apply Theorem 2.)

$$(19) \qquad X_{n+1} = \begin{cases} X_n + \log \gamma & \text{with prob } F_{X_n}(p_1)(1 - \pi_1) \\ & \qquad + (1 - F_{X_n}(p_1))(1 - \pi_2) = \varphi(X_n), \\ X_n + \log \beta & \text{with prob } [1 - \varphi(X_n)]. \end{cases}$$

Observe that

$$(20) \qquad \lim_{x \to +\infty} \varphi(x) = 1 - \pi_2 \quad \text{and} \quad \lim_{x \to -\infty} \varphi(x) = 1 - \pi_1 .$$

Combining (20) and Theorem 2, one then has immediately Theorem 4.

THEOREM 4. *For the contingent case of the two-operator model, let* $c = -\log \beta / \log (\gamma/\beta)$. *Then with probability one*

(i) *if* $1 - \pi_2 < c$ *and* $1 - \pi_1 > c$ *then*

$$\limsup_n p_n = 1 \quad \textit{and} \quad \liminf_n p_n = 0,$$

(ii) *if* $1 - \pi_2 < c$ *and* $1 - \pi_1 < c$ *then* $p_\infty = 1$,

(iii) *if* $1 - \pi_2 > c$ *and* $1 - \pi_1 > c$ *then* $p_\infty = 0$.

Moreover,

(iv) *if* $1 - \pi_2 > c$ *and* $1 - \pi_1 < c$ *then for some* δ *with* $0 < \delta < 1$

$$\Pr(p_n \to 1) = \delta, \qquad \Pr(p_n \to 0) = 1 - \delta.$$

The intuitive character of the distinction between the results expressed in (i) and (iv) of this theorem should be clear. If $1 - \pi_2 < c$ and $1 - \pi_1 > c$, then probability zero of an A_1 response and probability one of an A_1 response are both reflecting barriers, whereas if $1 - \pi_2 > c$ and $1 - \pi_1 < c$, they are both absorbing barriers.

It is also to be noticed that except when $1 - \pi_1 = c$ or $1 - \pi_2 = c$, Theorem 4 covers all values of β, γ, π_1, and π_2 for the contingent case. It can be shown [5] by deeper methods that if $1 - \pi_1 = c$ (or $1 - \pi_2 = c$) then probability one (respectively zero) of an A_1 response is again a reflecting barrier. These results agree with those given by Luce ([7], p. 124) and in addition settle most of the open questions in his Table 6. Detailed comparison is tedious because his classification of cases differs considerably from ours as given in the above theorem.

Contingent Reinforcement with Four Operators

We want finally to apply Theorem 3 to the contingent case of the general four-operator model formulated in (1). Analogous to (19),

$$(21)\qquad X_{n+1} = \begin{cases} X_n + \log \beta_{22} & \text{with prob } (1 - \pi_2)(1 - F_{X_n}(p_1)) = \varphi_{22}(X_n), \\ X_n + \log \beta_{12} & \text{with prob } (1 - \pi_1)F_{X_n}(p_1) = \varphi_{12}(X_n), \\ X_n + \log \beta_{21} & \text{with prob } \pi_2(1 - F_{X_n}(p_1)) = \varphi_{21}(X_n), \\ X_n + \log \beta_{11} & \text{with prob } \pi_1 F_{X_n}(p_1) = \varphi_{11}(X_n). \end{cases}$$

Also,

$$(22)\qquad \begin{cases} \lim_{x\to+\infty} \varphi_{22}(x) = 1 - \pi_2 , & \lim_{x\to-\infty} \varphi_{22} = 0, \\ \lim_{x\to+\infty} \varphi_{12}(x) = 0, & \lim_{x\to-\infty} \varphi_{12}(x) = 1 - \pi_1 , \\ \lim_{x\to+\infty} \varphi_{21}(x) = \pi_2 , & \lim_{x\to-\infty} \varphi_{21}(x) = 0, \\ \lim_{x\to+\infty} \varphi_{11}(x) = 0, & \lim_{x\to-\infty} \varphi_{11}(x) = \pi_1 . \end{cases}$$

Then

$$(23)\qquad \mu_+ = \sum_{j,k} \log \beta_{jk} \lim_{x\to+\infty} \varphi_{jk}(x) = \pi_2 \log \beta_{21} + (1 - \pi_2) \log \beta_{22} ,$$

and

$$(24)\qquad \mu_- = \sum_{j,k} \log \beta_{jk} \lim_{x\to-\infty} \varphi_{jk}(x) = \pi_1 \log \beta_{11} + (1 - \pi_1) \log \beta_{12} .$$

To apply Theorem 3 one also assumes that $\beta_{22} , \beta_{12} > 1 > \beta_{21} , \beta_{11} > 0$. On this assumption, and utilizing (23) and (24), we infer Theorem 5.

THEOREM 5. *For the contingent case of the four-operator model, with probability one*

(i) *if* $\mu_+ < 0$ *and* $\mu_- > 0$ *then* $\limsup_n p_n = 1$ *and* $\liminf_n p_n = 0$,

(ii) *if* $\mu_+ < 0$ *and* $\mu_- < 0$ *then* $p_\infty = 1$,

(iii) *if* $\mu_+ > 0$ *and* $\mu_- > 0$ *then* $p_\infty = 0$;

and if $\mu_+ > 0$ *and* $\mu_- < 0$, *then for some* δ *with* $0 < \delta < 1$

(iv) $\Pr(p_n \to 1) = \delta$, $\Pr(p_n \to 0) = 1 - \delta$.

Specialization of this theorem to cover the noncontingent case is immediate.

REFERENCES

[1] Bush, R. R., Galanter, E., and Luce, R. D. Tests of the "beta model." In R. R. Bush and W. K. Estes (Eds.), *Studies in mathematical learning theory*. Stanford: Stanford Univ. Press, 1959. Ch. 18.

[2] Chung, K. L. and Fuchs, W. H. J. On the distribution of values of sums of random variables. *Mem. Amer. Math. Soc.*, 1951, **6**, 1–12.

[3] Hodges, J. L. and Rosenblatt, M. Recurrence time moments in random walks. *Pac. J. Math.*, 1953, **3**, 127–136.

[4] Karlin, S. Some random walks arising in learning models I. *Pac. J. Math.*, 1953, **3**, 725–756.

[5] Lamperti, J. Criteria for the recurrence or transience of stochastic processes I. *J. math. Anal. Applications*, (in press).

[6] Lamperti, J. and Suppes, P. Chains of infinite order and their application to learning theory. *Pac. J. Math.*, 1959, **9**, 739–754.

[7] Luce, R. D. *Individual choice behavior*. New York: Wiley, 1959.

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1. "A Note on the Menger-Wieser Theory of Imputation," Hirofumi Uzawa, *Zeitschrift für Nationalökonomie*, Vol. XVIII, No. 3, 1958, pp. 318–34.
2. "On a Class of Capacitated Transportation Problems," Harvey M. Wagner, *Management Science*, Vol. 5, No. 3, April 1959, pp. 304–18.
3. "The Case for 'Revealed Preference'," Harvey M. Wagner, *The Review of Economic Studies*, Vol. XXVI, No. 3, June 1959, pp. 178–89.
4. "Chains of Infinite Order and Their Application to Learning Theory," John Lamperti and Patrick Suppes, *Pacific Journal of Mathematics*, Vol. 9, No. 3, 1959, pp. 739–54.
5. "Competitive Stability Under Weak Gross Substitutability: The 'Euclidean Distance' Approach," Kenneth J. Arrow and Leonid Hurwicz, *International Economic Review*, Vol. 1, No. 1, January 1960, pp. 38–49.
6. "Prices of the Factors of Production in International Trade," Hirofumi Uzawa, *Econometrica*, Vol. 27, 3, July 1959, pp. 448–68.
7. "Stability and Non-Negativity in a Walrasian Tâtonnement Process," Hukukane Nikaidô and Hirofumi Uzawa, *International Economic Review*, Vol. 1, No. 1, January 1960, pp. 50–59.
8. "The Capacity Method of Quadratic Programming," H. S. Houthakker, *Econometrica*, Vol. 28, 1, January 1960, pp. 62–87.
9. "Optimization, Decentralization, and Internal Pricing in Business Firms," Kenneth J. Arrow, Stanford University, *Contributions to Scientific Research in Management*, pp. 9–18.
10. "Classification Procedures Based on Dichotomous Response Vectors," Herbert Solomon, Stanford University, *Contributions to Probability and Statistics*, 1960, pp. 414–23. Stanford University Press.
11. "The Work of Ragnar Frisch, Econometrician," K. J. Arrow, *Econometrica*, Vol. 28, 2, April 1960, pp. 175–92.
12. "Best Linear Index Numbers of Prices and Quantities," H. Theil, *Econometrica*, Vol. 28, 2, April 1960, pp. 464–80.
13. "Walras' Tâtonnement in the Theory of Exchange," H. Uzawa, *The Review of Economic Studies*, Vol. XXVII, No. 3, pp. 182–94.
14. "Stability of the Gradient Process in n-Person Games," K. J. Arrow and Leonid Hurwicz, *J. Soc. Indust. Appl. Math.*, Vol. 8, No. 2, June 1960.
15. "Decision Theory and the Choice of a Level of Significance for the t-test," K. J. Arrow, Stanford University, *Contributions to Probability and Statistics*, 1960, pp. 70–78. Stanford University Press.
16. "Some Asymptotic Properties of Luce's Beta Learning Model," John Lamperti and Patrick Suppes, *Psychometrika*, Vol. 25, No. 3, September 1960, pp. 233–41.