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2 Some Foundational Problems in the Theory of Visual Space

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ABSTRACT

The most important general feature of visual space is that it is context dependent, a characteristic of physical systems rather than classical geometrical ones Using results from some classical experiments by Foley and by Wagner, arguments are given to show that no reasonably simple unified set of axioms in the spirit of qualitative synthetic geometry can be given for the structure of visual space.

PRELIMINARIES

Perhaps the most important feature of visual space that must be taken account of in a thorough analysis is that visual space is more like a physical system than a geometrical one. What I mean by this is that there are strong context affects which do not exist at all in classical geometry. The motion of two particles in classical mechanics is very much affected by bringing into close proximity a third particle, but the geometric relation at a given time of these two particles is not affected by the presence or absence of this third particle. On the other hand, perceptual judgments of symmetry or of congruence are known from many experiments to be much affected by context. This is a strong warning from the very beginning that we cannot hope to have the theory of visual space placed too firmly and thoroughly within the framework of classical geometry.

The second point that follows from such strong contextual effects is that multiple geometrical models of visual space should be required because different contexts will lead to different models. The geometric relations will not be the same from context to context. Once we recognize this aspect then what we can hope for, insofar as we are attempting to obtain quantitative results, is to find models that are parametrically stable in a given kind of context. But even this hope is too optimistic, as we shall see shortly.

SPACES OF CONSTANT CURVATURE

The Luneburg quantitative approach to visual space is perhaps still the best theoretically developed program, even now after nearly half a century since it was first introduced. In orthogonal sensory coordinates the line element *ds* can be represented in terms of sensory coordinates α, β and γ (these Greek letters are not the ones used by Luneburg) by

$$ds^2 = \frac{d\alpha^2 + d\beta^2 + d\gamma^2}{[(1 + \frac{1}{4}K(\alpha^2 + \beta^2 + \gamma^2)]^2}, \qquad (2.1)$$

where
$K = 0$ for Euclidean space,
$K < 0$ for hyperbolic space,
$K > 0$ for elliptic space.

The best recent treatment is Indow (1979).

It is important to recognize that the Luneburg approach is strictly a psychophysical approach to visual space. It assumes a transformation of physical Euclidean space to yield a psychological one. In this sense it is not a measurement-theoretic approach or a qualitative approach to the axioms of visual space.

By far the best experimental studies of the Luneburg ideas are those that have been conducted by Indow and his colleagues over many years, although a large number of other investigators have also contributed. A review of this extensive literature is to be found in Volume II of *Foundations of Measurement* (Suppes, Krantz, Luce, & Tversky, 1989). It is also the case that the best analysis of multidimensional scaling of spaces of constant curvature has been given by Indow (1974, 1982).

As I want to make clear from these references, I myself have learned more about the theory of visual space from Indow and his colleagues than from anyone else. I was initially enormously skeptical of the Luneburg ideas, and I came to realize they could be converted into a realistic program just because of the extraordinary, careful experiments performed by Indow and his colleagues. In this case the fact that the program has not turned out to be more satisfactory than it is is not because of the weakness of the experiments but in fact because of their very strength. They have given us confidence to recognize that there are fundamental things wrong with the Luneburg approach to visual space. Above all, as Indow and his colleagues have brought out on several occasions, there is a complete lack of parametric stability once we assume that a space of constant curvature, for example, negative curvature in the hyperbolic case, is a reasonable hypothesis. When we estimate the curvature we find that even for the same experiments the results are not stable from day to day for a given subject, and certainly when we transfer from one kind of experiment, for example, the classical alley experiments, to judgments of a different kind, there is little transfer at all of the parametric values estimated in one situation to the next.

TWO IMPORTANT COUNTEREXAMPLES

I now want to turn to two important experimental results based upon qualitative considerations and that fly very much in the face of the Luneburg ideas, which fundamentally assume that visual space is a space of constant curvature as defined by Equation 2.1. In fact these counterexamples are such that they raise problems that I am not able to answer about how we should think about the geometry of visual space even in a quite simple range of experiments.

The first is a beautiful counterexample due to Foley (1972). The situation is shown in Figure 2.1. The instructions to the subject are to make judgments of perpendicularity ($\perp$) and congruence ($\simeq$) as follows, with the subject given point A as fixed in the depth axis directly in front of his position as observer O

1. Find point B so that $AB \perp OB$ & $AB \simeq OB$.
2. Find point C so that $OC \perp OB$ & $OC \simeq OB$.
3. Judge the relative lengths of OA & BC.

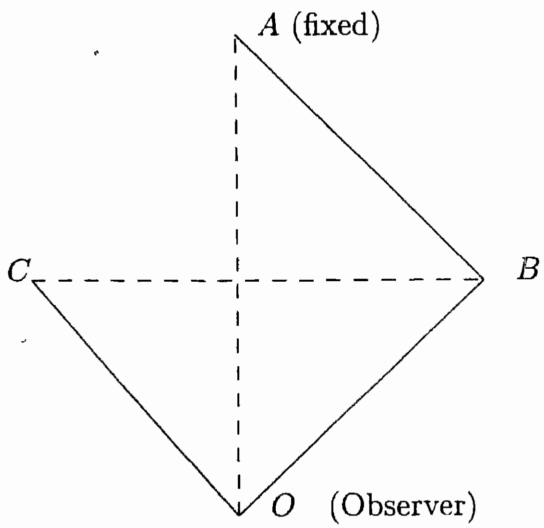

FIG. 2.1. Foley experiment.

The results are that 24 subjects in 40 of 48 trials judged BC significantly longer than OA. This judgment that BC is longer than OA contradicts properties of any space of constant curvature, whether that curvature is positive, negative, or zero. This simple experiment of Foley pushes us at once completely outside the framework of spaces of constant curvature and leads us to abandon for detailed purposes any hope of fully satisfying the Luneburg psychophysical postulates.

The second experiment is Wagner (1985), which dealt with perceptual judgments about distances and other measures among 13 white stakes in an outdoor viewing area. In physical coordinates let x equal measurement along the depth axis and y along the frontal axis, and let perceptual judgments of distance be shown by primes. Then the general result of Wagner is that $x' \simeq 0.5y'$ if physically $x = y$. Notice how drastic the foreshortening is along the depth axis. This result of Wagner's is not anomalous or peculiar but represents a large number of different perception experiments showing dramatic foreshortening along the depth axis.

PARTIAL AXIOMS FOR THE FOLEY AND WAGNER EXPERIMENTS

I am not able to give a fully satisfactory geometric analysis of the strongly supported experimental facts found in these experiments, but I do think there are some things to be said of interest that will help clarify the foundational situation. I have divided the axioms that I propose into various groups.

1. Affine Plane. These I take to be standard axioms. We would of course, reduce them and not require the whole plane but that is not important here. We can take as primitives either betweenness or parallelness and some concept like that of a midpoint algebra. Again the decision is not critical for considerations here. The reader can think just in terms of betweenness as an awkward but thoroughly developed theory of ordered affine planes. Let B be the ternary relation of betweenness. I use the simple and suggestive following notation for this relation. For any three points $a, b, c, B(a, b, c)$ if and only if $a|b|c$; i.e., point b is between points a and c, with weak inequality intended. In other words, if φ is a real-valued function on the set of points, the intended numerical interpretation is

$$a|b|c \quad \text{iff} \quad \varphi(a) \leq \varphi(b) \leq \varphi(c) \quad \text{or} \quad \varphi(c) \leq \varphi(b) \leq \varphi(a).$$

Axioms on betweenness needed to characterize an affine plane are given in the Appendix.

We now add judgments of perceived congruence ($\approx$).

2. Three Distinguished Points. Intuitively, let o_1 be the center of the left eye, o_2 the center of the right eye, and o the bisector of the segment o_1, o_2. Explicitly, they satisfy the following two axioms.

2a. The three points are collinear and distinct;

2b. $o_1 o \simeq o o_2$.

3. Axioms of Congruence

3a. Opposite sides of any parallelogram are congruent

3b. If $aa \simeq bc$, then $b = c$.

3c. $ab \simeq ba$.

3d. If $ab \simeq cd$ & $ab \simeq ef$, then $cd \simeq ef$.

3e. If $a|b|c$ & $a'|b'|c'$, $ab \simeq a'b'$ & $bc \simeq b'c'$, then $ac \simeq a'c'$ (this is a familiar and weak additivity axiom).

The *frontal axis* is the line containing o_1, o, and o_2 and the *depth axis* is the half-line through o such that for any point a on the axis $o_1 a \simeq o_2 a$. (Notice that we cannot characterize the depth axis in terms of a general notion of perpendicularity for that is not available, and in fact will not be available within the framework of these axioms. The depth axis is only a half-line because a subject cannot see directly behind the frontal axis.)

3f. *First special congruence axiom.* If $a \neq c$, a, c on frontal axis and b is on the depth axis, and $ao \simeq oc$, then $ab \simeq bc$ (see Figure 2.2).

3g. *Second special congruence axiom.* If $a \neq c$, a, c on the frontal axis, $ao \simeq oc$, ab and cd $\|$ to depth axis, $ab \simeq cd$, then $ob \simeq od$ (see Figure 2.3).

The last two special axioms of congruence extend affine congruence to congruence of segments that are not parallel, but only in the case where the segments are symmetric about the depth axis as is seen from Figures 2.2 and 2.3. This means that we have a weak extension of affine congruence. An extension that is far too weak even to give us the axioms of congruence for absolute spaces (see *Foundations of Measurement II*, Chapter 13). (Absolute geometry can be thought of this way. Drop the Euclidean axiom that through a given point a exterior to a line α there is at most one line through a that is parallel to α and lies in the plane formed by a and α. Adding this axiom to the axioms of absolute geometry gives us Euclidean geometry, as is obvious. What is much more interesting is that adding the negation of this axiom to those of absolute geometry gives us hyperbolic geometry.)

We can prove the following theorem.

Theorem 2.1. Let the half-space consisting of all points on the same side of the frontal axis as the depth axis be designated the frontal half-plane:

(1) The frontal half-plane is isomorphic under a real-valued function φ to a two-dimensional affine half-plane over the field of real numbers, with the x-axis the depth axis and the y-axis the frontal axis. Moreover, congru-

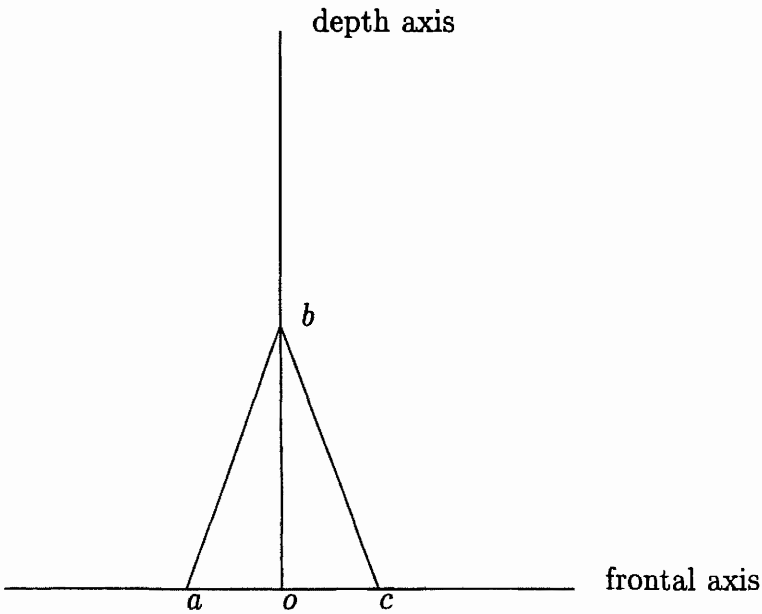

FIG. 2.2. Congruence Axiom 3f.

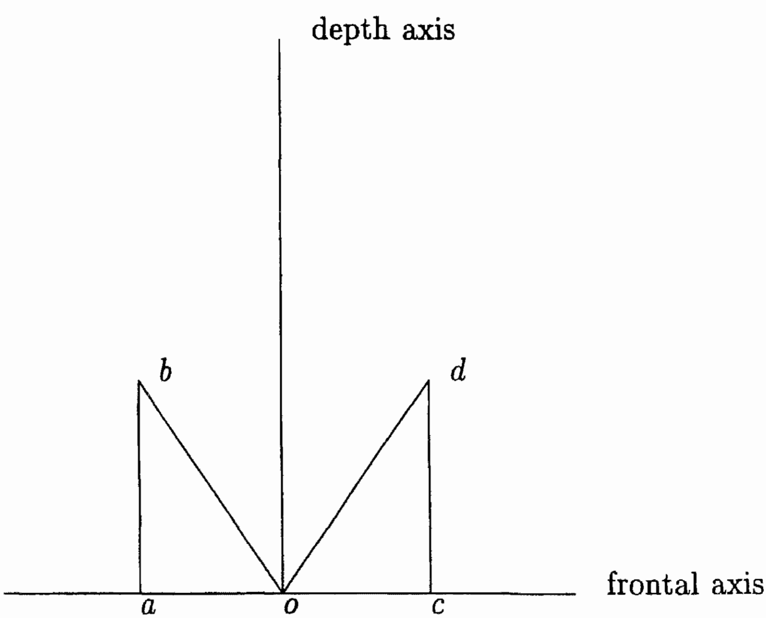

FIG. 2.3. Congruence Axiom 3g.

ence as measured by the Euclidean metric is satisfied when line segments are congruent; i.e., if ab $\simeq$ cd then $\Sigma_{i=1}^{2}(\varphi_o(a) - \varphi_i(b))^2 = \Sigma_{i=1}^{2}(\varphi_i(c) - \varphi_i(d))^2$.

(2) The only affine transformations possible are those consisting of stretches α of the frontal axis and stretches β of the depth axis with α, $\beta > 0$.

The proof of (1) is obvious from familiar results of classical geometry. The proof of (2) follows from observing that the affine transformations described are the only ones that preserve the symmetry around the depth axis required by the two special congruence axioms.

It is clear that the results of Theorem 2.1 are quite weak. The theorem is not contradicted by the Foley and Wagner experiments, but this is not surprising, for the apparatus of betweenness plus symmetric congruence about the depth axis cannot describe the results of either experiment. If we were to add a concept of perpendicularity, as required by the Foley procedure, then we would essentially get a Euclidean half-plane, and the resulting structure would contradict the Foley results.

Correspondingly, we cannot describe the Wagner psychophysical results of extensive perceptual foreshortening along the depth axis without adding some psychophysical assumptions radically different from those of Luneburg. Of course, the Foley results also are best interpreted as perceptual foreshortening along the depth axis. The natural conclusion is that we cannot consistently describe visual geometric relations in any space close to the specificity and simplicity of structure of a space of constant curvature.

Even the weak affine structure of Theorem 2.1 is too strong and probably should be replaced by a standard version of absolute geometry, but with the congruence axioms weakened as given above. Such an absolute geometry can be extended to hyperbolic geometry, but the affine structure cannot. What we seem to end up with is a variety of fragments of geometric structures to describe different experiments. A hyperbolic fragment perhaps for alley experiments, a fragment outside the standard ones for the Foley experiments, etc.

The goal of having a unified structure of visual space adequate to account for all the important experimental results now seems mistaken. A pluralistic and fragmentary approach seems required.

APPENDIX

Let A be a nonempty set, and let B be a ternary relation of betweenness as discussed in the text. Then we have the following definition, where a *line ab* is the set of all points c such that $a|c|b$ or $c|b|a$ or $c|b|a$

Definition 2.1. A structure (A, B) is an *affine plane* if and only if the following axioms are satisfied for a, b, c, d, e, f, g, a′, b′ and c′ in A:

1. If a|b|a, then a = b.
2. If a|b|c, then c|b|a.
3. If a|b|c and b|d|c, then a|b|d.
4. If a|b|c and b|c|d and b ≠ c, then a|b|d.
5. (Connectivity) If a|b|c, a|b|d, and a ≠ b, then b|c|d or b|d|c.
6. (Extension) There exists f in A such that f ≠ b and a|b|f.
7. (Pasch's Axiom) If abc is a triangle, b|c|d, and c|e|a, then there is on line de a point f such that a|f|b.
8. Desargues's axiom. If d|a|a′, d|b|b′, d|c|c′, a|b|e, a′|b′|e, a|c|f, a′|c′|f, b|c|g, b′|c′|g, not d|a|b, not a|b|d, not b|d|a, not d|b|c, not b|c|d, not c|d|b, not d|c|a, not c|a|d, not a|d|c, and a ≠ a′, then e|f|g.
9. (Axiom of Completeness) For every partition of a line into two non-empty sets Y and Z such that
   (i) no point b of Y lies between any a and c of Z and
   (ii) no point b′ of Z lies between any a′ and c′ of Y,
   there is a point b of Y ∪ Z such that for every a in Y and c in Z, b lies between a and c.
10. (Dimensionality). There are three noncollinear points a_0, b_0, c_0 in A such that for any point d′ in A there are points e′ and f′ such that e′ ≠ f′, e′ and f′ lie each on one of the three lines a_0b_0, a_0c_0, and b_0c_0, and d′, e′, and f′ are collinear.

The formulations of Axioms 8 and 10 just in terms of betweenness are rather complicated. But Foley (1964), in an excellent experimental study, found that a number of subjects made judgments satisfying the Desarguesian property (Axiom 8).

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