26

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SOME OPEN PROBLEMS IN THE FOUNDATIONS OF SUBJECTIVE PROBABILITY[1]

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Not only am I the only speaker who is a philosopher, but I am probably the only person attending this conference who is a philosopher; thus I should be expected to give you some words of wisdom. But I really do not have any such words to say. More particularly, I do not want to offer any general defense of subjective probability, or the meaning of subjective probability. I do not mean to admit by this that I am unwilling to offer such a defense. It is just that I do not want to rehash an old story. This morning I am going to talk about more limited problems than a general defense of the meaning and possible applications of notions of subjective probability. Secondly, in talking about problems of subjective probability, I will talk about some problems which interest me. I will not maintain that these problems are the most important, or the most interesting to everyone - they are problems which have interested me. Thirdly, I will be talking in the framework, particularly in the first part of the talk, that Savage introduced in his book, <u>Foundations of Statistics</u>.

The kind of model introduced in that book is as follows: there is a set S of states of nature, a set C of consequences, and a set D of decisions or acts which are functions mapping S into C. The decision-maker's problem is to choose from the decisions or acts that are available one which is in some sense optimal. The analysis which Savage's book leads to is the standard MEU behavioral pattern (maximization of expected utility). Savage introduces seven axioms in terms of an ordering relation $\precsim$ on acts or decisions. For example, Axiom 1 asserts that this relation is transitive and connected. By connected I mean we can weakly choose between any two acts. Naturally though, this axiom does not take us very far. The upshot of the six additional axioms is to yield the MEU result; namely, that if the postulates, in terms of this relation on acts, are satisfied, then we can show that in choosing an act from the set available, a

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person is maximizing expected utility. We mean by this that the person has a utility function on the set of consequences and a subjective probability distribution on the set of states, and the expectancy is with respect to this subjective probability distribution on the set of states.

This kind of maximization of expected utility behavior is not a notion which in any sense originates with Savage; it is very old - in fact it goes back to James Bernoulli in the 18th century. Within this kind of framework there are two major classes of problems that I would like to discuss. The first class of problems, in a certain definite sense, is oriented toward normative behavior, i.e., telling a person how he should behave. The second class of problems is oriented toward a descriptive application. To what extent can we use notions of utility and subjective probability to discuss or to analyze the actual behavior of people? Under the normative heading I will be particularly interested in what I will call problems of axiomatisability and definability, and under the second general heading in what I call behavioristic problems. So let me now address myself to problems of axiomatizability and definability. I want to discuss certain axiomatizability problems that we can raise and which seem to be interesting and somewhat difficult to solve. In discussing these axiomatizability problems there will be some notions perhaps not completely familiar. I will try to indicate intuitively the character of the results, even if I do not explicate all the technical details.

1. Constant acts. A problem which arises immediately in the Savage framework is that of the constant functions or constant acts. By a constant act I mean one that yields the same consequence whatever the state of nature. In more formal terms, a constant act is a function in the set D whose value is the same for all arguments; that is, for all states of nature. Savage's analysis requires that D include the set of all constant acts. An earlier unpublished paper of Herman Rubin's, which assumes some quantitative postulates but is concerned with deriving the existence of a Bayesian distribution on the states of nature, also requires such acts. My own set of axioms [1], analogous to Savage's but more closely related to the approach (1926) of Frank Ramsey to these problems, demands inclusion of the constant acts.

I know of no analysis which does not require these acts, and yet I want to show by analysing an example of Savage's just how difficult it is to interpret them. Suppose a man is mixing a six-egg omelet and has put five eggs in a bowl, the problem is what to do with the sixth egg. (For some reason he has a suspicion it may be rotten.) For the moment, we will reduce the problem to two acts - Act I, put egg in a separate bowl; Act II, put egg directly in with other five. The states of nature are S_1 - the egg is rotten, S_2 - the egg is fresh. There are two possibilities. If he puts the egg in a separate bowl and the egg is rotten then he can replace it. He does not ruin the omelet. If he puts the egg in with the other five and the egg is rotten, he ruins all six. I will assume it is very difficult to separate out the rotten egg when it is mixed in with five good ones. On the other hand, it is troublesome and time-consuming to put the egg in a separate bowl. If the man strongly believes the egg is fresh, he is very likely to put it directly into the bowl containing the five other eggs. The constant acts now enter in the following way. In order to prove that the axioms of behavior yield an MEU result, it is necessary (but not sufficient in this case) to extend our set of acts to include the constant acts. In particular, we need to have an act which, even if the egg is fresh, leads to a consequence of ruining the omelet. In other words, totally unrealizable acts are required in order to derive the MEU result. We can certainly, introspectively in some general way, understand what these acts mean. We cannot realize them. To my mind, it is a severe weakness of a theory which claims to be behavioristic to have such acts inextricably included in its formal setup; they hark back all too much to the verbalistic tradition which Savage has so admirably criticized. It is, of course, not playing the game to adopt some ad hoc device like that of a random mechanism whose workings do not affect and are unaffected by goings on in the rest of the world. The assumption of such a mechanism is a patent <u>deus ex machina</u> and nullifies one of the primary aims of the Savage kind of analysis; namely, to extend the theory of rational behavior to areas of action where it is unnatural to think in terms of random mechanisms.

2. <u>Theory of pure rationality</u>. The axioms of the various systems of rational behavior which have been proposed by Ramsey, de Finetti, Savage, and others, including myself, may be divided into two classes. In the first class go those which may be thought of as holding anywhere and anytime. These I call <u>pure</u> axioms of rationality. An example of a pure axiom is the postulate that the preference relation on the set of acts is transitive. In the second class belong those which postulate some special struc-

tural property of the environment and possibly of the decision maker. These I call structural axioms. The main structural axiom in Savage's setup is, roughly speaking, that the decision-maker can partition the set of states of nature as fine as he pleases in terms of probability. The result of this axiom is that there must be, in any model satisfying Savage's axioms an infinity of states of nature, and given any probability ϵ no matter how small there is a set of states which has a probability no greater than ϵ. Such a requirement has nothing in itself to do with the concept of pure rationality, that is, with the concept of making a rational decision. I consider it a structural imposition, a limitation on the range of applicability of the theory.

Savage's axiom is, of course, not the only kind of structural assumption which can be made. In my Berkeley Symposium paper, the number of states of nature is arbitrary and I depended on a different kind of structural axiom; namely, that between any two consequences the decision-maker can find another which is equally spaced in utility between them. This axiom implies that, except in the trivial case of all consequences being equally prized, there must be an infinity of consequences. In another paper [2], Donald Davidson and I used the structural assumption that there are only a finite number of consequences which are equally spaced in utility.

Two things about these structural axioms should be clear. In the first place, although I have used quantitative or semi-quantitative language in formulating them, all of them may be formulated in terms of very primitive and qualitative concepts. Secondly, in all systems of axioms formulated within the Savage kind of framework with which I am familiar, such axioms are necessary to prove the MEU kind of result. And now I want to give some relatively fundamental reasons for this necessity.

To begin with it will be desirable to have a more exact definition of the notion of a pure axiom of rationality. I say that an axiom of behavior is a pure axiom if and only whenever it is satisfied in a model $\mathfrak{M}$ it is satisfied in any submodel of M. Consider, for instance, the axiom that the preference relation $\preccurlyeq$ on the set D of acts is transitive. Any ordered couple $\mathfrak{U} = \langle A, R\rangle$ is a possible realization of this axiom if A is a non-empty set and R is a binary relation on A. A possible realization $\mathfrak{U}$ is a model of the axiom if the relation R is transitive on A. A model $\mathfrak{U}' = \langle A', R'\rangle$ is a submodel of the model $\mathfrak{U}$ if A' is a subset of A and R' is the relation R restricted to the set A'. It is easily verified that any submodel of a model of the transitivity axiom is also a model of the axiom, and consequently this axiom is pure. It may also easily be shown that the connectivity axiom for the preference relation $\precsim$ is also a pure axiom. Suppose now we consider an axiom which says three things: (i) the preference relation is transitive on the set of acts; (ii) it is also coneected on this set; and (iii) there is one act which is (weakly) preferred to all others, that is, there is an act d_1 such that for all acts d_2, $d_2 \precsim d_1$. Now this axiom is pure if we restrict ourselves to finite models because any *finite model* having properties (i) and (ii) will also have (iii). However, if we permit infinite models, then the axiom is no longer pure, because an infinite set which has a greatest element with respect to an ordering relation may have infinite subsets which do not have such an element. For example, the set of all rational numbers x such that $0 \leq x \leq 1$ has 1 as its greatest element with respect to the natural ordering $\leq$, but the subset of numbers such that $0 \leq x < 1$ has *no* such greatest element. This axiom may suggest that structural axioms are always existential in character, but this is not always the case; for instance, the one Davidson and I used [2] is not existential in form.

The question I now pose is this. What are the possibilities of axiomatizing the theory of pure rationality? In the first place, it is reasonable to restrict ourselves to recursive axiomatizations. A recursive axiomatization of a subject may consist of an infinite list of axioms, but there is a mechanical method for deciding whether or not a statement is an axiom. A simple example of a non-recursive axiomatization may be given for arithmetic, namely the single sentence "A statement S of arithmetic is an axiom if and only if it is true." This axiomatization is non-recursive because it follows from fundamental results of Gödel and Tarski that there is no mechanical method for deciding whether or not a sentence of arithmetic is true.

Secondly, I shall restrict consideration to what are called in logic first-order axioms; that is, we shall permit the variables which occur in the axioms to take as values only the elements of the set D of acts. This is a strong restriction, for it prohibits, for example, any Archimedean axiom which uses an integer-valued variable. The reason for this restriction is that I want to discuss some negative results of a metamathematical or logical character. The difficulties of obtaining any general results on problems of axiomatizability when the axioms are not first-order are considerable. Having imposed the restriction of first-order axioms, it will be necessary to consider only finite models, for it is well-known that if a set of first-order axioms has an infinite model it has an infinite number of models of different infinite cardinality. Consequently it is impossible to give for infinite models first-order axioms on the basis of which the existence of numerical utility or subjective probability functions may be established.

Thirdly, for purposes of simplicity it will be desirable to deal with a situation which permits only two states of nature s_1 and s_2 with equal subjective probabilities. Thus $\sigma(s_1) = \sigma(s_2)$, where $\sigma(s_i)$ is the numerical subjective probability of state s_i. And in terms of expected utility we may then write for d_1, d_2 in D: $d_1 \preccurlyeq d_2$ if and only if

(1) $$\sigma(s_1)u(d_1(s_1)) + \sigma(s_2)u(d_1(s_2)) \leq \sigma(s_1)u(d_2(s_1)) + \sigma(s_2)u(d_2(s_2)),$$

where u is the numerical utility function of the set C of consequences. Now since $\sigma(s_1) = \sigma(s_2)$, we have equivalent to (1)

(2) $$u(d_1(s_1)) + u(d_1(s_2)) \leq u(d_2(s_1)) + u(d_2(s_2)),$$

which in turn is equivalent to

(3) $$u(d_1(s_1)) - u(d_2(s_1)) \leq u(d_2(s_2)) - u(d_1(s_2)).$$

Whence the theory of pure rationality for this situation of two states of nature with equal probability reduces to axiomatizing the quaternary relation R on the set C of consequences such that there is a numerical function u on C with the property that for every x, y, z, and w in C, x y R z w if and only if

(4) $$u(x) - u(y) \leq u(z) - u(w).$$

The transformation from the relation $\preccurlyeq$ on D to the relation R on C is made for technical purposes. Several years ago I thought it would not be a difficult matter to axiomatize R in terms of a finite list of first-order sentences so as to satisfy (4). The problem has not only proved difficult, but in fact Dana Scott and I have shown that it cannot be axiomatized by a finite list of first-order axioms none of which is existential in character [3]. Intuitively it seems that existential sentences cannot offer any real help when it is required that the axioms be closed under submodels, but we have been unable to back up this intuition with a formal proof. So even for this simple case, the problem of finite axiomatization is not settled.

It is possible to give a recursive axiomatization of the relation R (for finite models) by enumeration of what are technically called the isomorphism types of R. We start with sets of cardinality one and list the single isomorphism type, and proceed in this way for each finite cardinal n, listing the types in some fixed order. The difficulty, of course, is that this kind of recursive axiomatization is intuitively completely uninformative. This is by no means always the case with recursive axiomatizations of a theory, as the standard axioms for elementary number theory or those for Zermelo set theory adequately testify. The negative proof given by Scott and me [3] depended upon showing that an infinite but recursive list of axioms which permitted "addition" of intervals is in a certain sense necessary, and at one time we thought a reasonably satisfactory recursive axiomatization could be given which used this addition schema and a finite number of additional axioms. Unfortunately Robert McNaughton produced a counterexample to this system of axioms. His counterexample consists of a set of twenty-two elements; it satisfies the addition schema but does not permit a numerical representation of the kind characterized by (4). It seems that the problem of finding a reasonably appealing recursive axiomatization is difficult.

A fortiori these problems of axiomatization are unsolved for models which permit more states of nature.

3. Behavioristic foundations of subjective probability and utility. From a psychological standpoint the most undesirable thing about the MEU result within the Savage kind of framework is its static character. There is no attempt to explain how an organism comes to have subjective degrees of beliefs about possible states of nature, or evaluations of the relative desirability of different possible consequences. There is no theory as to how the environment interacts with the individual.

I have recently derived from the general assumptions of stimulus learning theory a utility for some simple choice situations [4]. I want briefly to describe these results and then to indicate some of the open problems. Stimulus sampling learning theory was first given a quantitative formulation in 1950 by the psychologist W.K. Estes, and has since been developed by a number of investigators. The basic ideas run as follows. The organism is presented with a sequence of trials on each of which he makes a response that is one of several possible choices. In any particular setup it is assumed that there is a set of stimuli from which the organism draws a sample at the beginning of each trial. It is assumed that on each trial each stimulus is conditioned to exactly one response. The probability of making a given response on any trial is postulated to be simply the proportion of sampled stimuli which are conditioned to that response.

Learning takes place by the following mechanism. At the end of a trial a reinforcing event occurs which identifies that one of the possible responses which was correct. The sampled stimuli become conditioned to this response, and the organism begins another trial in a new state of conditioning.

Naturally this account of stimulus sampling theory is a highly simplified one, and yet it should be clear in what sense this theory is dynamic rather than static, and thus provides a theoretical analysis of how the organism is interacting with its environment.

The kind of utility results obtained from this theory thus far are easily sketched. Suppose a person is on each trial presented with one of several pairs of slot machines. That is, on each trial he chooses which of two slot machines to play, but the pairs presented vary from trial to trial. (When there are exactly two slot machines, this is the familiar two-armed bandit problem.) Let there be N slot machines with π_i the probability of payoff of the i^{th} machine (the probability π_i is not known to the person.) Then the following utility function satisfying a requirement like (4) may be derived from stimulus sampling theory:

$$u(i) = \log \frac{1}{\epsilon_i(1 - \pi_i)} ,$$

where ϵ_i is the learning parameter associated with the i^{th} machine.

It is still far from clear how this kind of result may be extended to more complicated behavioral situations. Moreover, it is not yet clear how both subjective probability and utility functions may be derived from stimulus sampling theory even for very simple situations. Positive solution of these problems would provide yet another stepping stone toward the construction of a psychologically sophisticated theory of actual inductive behavior.

BIBLIOGRAPHY

[1] Suppes, P., "The Role of Subjective Probability and Utility in Decision-Making," Proc. of the Third Berkeley Symposium on Mathematical Statistics and Probability (1955), pp. 61-73.

[2] Davidson, D., and Suppes, P., "A Finitistic Axiomatization of Subjective Probability and Utility," Econometrica, Vol. 24 (1956), pp. 264-275.

[3] Scott, and Suppes, P., "Foundational Aspects of Theories of Measurement," Journal of Symbolic Logic, Vol. 23 (1958), pp. 113-128.

[4] Suppes, P., "Behavioristic Foundations of Utility," Technical Report No. 23, Institute for Mathematical Studies in the Social Sciences, Applied Mathematics and Statistics Laboratories, Stanford University, Contract NR 171-034, 1959.