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Stimulus-Sampling Theory for a Continuum of Responses

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1. Introduction

The aim of the present investigation is to extend stimulus-sampling theory to situations involving a continuum of possible responses. The theory for a finite number of responses stems from the basic paper by Estes [2]; the present formulation will resemble most closely that given for the finite case in [4]. In a previous study [3] I was concerned with a corresponding extension of linear learning models, and several results of that study are, as we shall see, closely related to the present one.

The experimental situation consists of a sequence of trials. On each trial the subject (of the experiment) makes a response from a continuum of possible responses; his response is followed by a reinforcing event indicating the correct response for that trial. In situations of simple learning, which are characterized by a constant stimulating situation, responses and reinforcements constitute the only observable data, but stimulus-sampling theory postulates a considerably more complicated process which involves the conditioning and sampling of stimuli. In the finite case the usual assumption is that on any trial each stimulus is conditioned to exactly one response. Such a highly discontinuous assumption seems inappropriate for a continuum of responses, and I have replaced it with the postulate that the conditioning of each stimulus is *smeared* over a certain interval of responses, possibly the whole continuum. In these terms, the conditioning of any stimulus may be represented uniquely by a *smearing distribution*. These distributions, one for each stimulus, will play the same role as did the single smearing distribution introduced in my earlier paper on linear models [3].

The theoretically assumed sequence of events on any trial may then be described as follows:

trial begins with each stimulus in a certain state of conditioning	→	certain stimuli are sampled	→	response occurs	→	reinforcement occurs	→	possible change in conditioning occurs.

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The sequence of events just described is, in broad terms, postulated to be the same for finite and infinite sets of possible responses. Differences of detail will become clear. The main point of the axioms in Section 2 is to state specific hypotheses about this sequence of events. As has already been more or less indicated, three kinds of axioms are needed: conditioning axioms, sampling axioms, and response axioms.

Section 3 contains some general theorems of the theory. Section 4 considers in some detail the classical case of non-contingent reinforcement. Section 5 treats of other cases more superficially.

Although no experimental data will be described in this paper, it will perhaps help to describe schematically one piece of apparatus which has been used to test the theory extensively. The subject is seated facing a large circular vertical disc. He is told that his task on each trial is to predict by means of a pointer where a spot of light will appear on the rim of the disc. The subject's pointer predictions are his responses in the sense of the theory. At the end of each trial the "correct" position of the spot is shown to the subject, which is the reinforcing event for that trial. The most important variable controlled by the experimenter is the choice of a particular probability distribution of reinforcement.

2. Axioms

The axioms are formulated verbally but with some effort to convey a sense of formal precision. It is not difficult, although not wholly routine, to convert them into a mathematically exact form. As already indicated, they fall naturally into three groups. In the statement of the axioms we use x for the response variable and z for the parameter of the smearing distribution $K_s(x; z)$ of any stimulus s. Moreover, z is the mode of the distribution; for the circular disc apparatus it is also assumed to be the mean, but not all apparatus to which the theory applies is so completely symmetric.

CONDITIONING AXIOMS

C1. *For each stimulus s there is on every trial a unique smearing distribution $K_s(x; z)$ on the interval $[a, b]$ of possible responses such that* (a) *the distribution $K_s(x; z)$ is determined by its mode z and its variance*; (b) *the variance is constant over trials for a fixed stimulating situation*; (c) *the distribution $K_s(x; z)$ is continuous and piecewise differentiable in both variables.*

C2. *If a stimulus is sampled on a trial, the mode of its smearing distribution becomes, with probability θ, the point of the response (if any) which is reinforced on that trial; with probability $1-\theta$ the mode remains unchanged.*

C3. *If no reinforcement occurs on a trial, there is no change in the smearing distributions of sampled stimuli.*

C4. *Stimuli which are not sampled on a given trial do not change their smearing distributions on that trial.*

C5. *The probability θ that the mode of the smearing distribution of a* *sampled stimulus will become the point of the reinforced response is independent of the trial number and the preceding pattern of occurrence of events.*

SAMPLING AXIOMS

S1. *Exactly one stimulus is sampled on each trial.*

S2. *Given the set of stimuli available for sampling on a given trial, the probability of sampling a given element is independent of the trial number and the preceding pattern of occurrence of events.*

RESPONSE AXIOMS

R1. *If the sampled stimulus s and the mode z of its smearing distribution are given, then the probability of a response in the interval* $[a_1, a_2]$ *is* $K_s(a_2; z) - K_s(a_1; z)$.

R2. *This probability of response is independent of the trial number and the preceding pattern of occurrence of events.*

Because of the similarity of these axioms to those in Suppes and Atkinson [4], I shall here mainly comment on those aspects peculiar to the continuum case. In the finite case the complicated form of Axiom C1 reduces simply to the assertion that on any trial each stimulus is conditioned to exactly one response. As already remarked, the assumption [C1(a)] that the smearing distribution of any stimulus is determined by its mode and variance, rather than its mean and variance, is used in order to permit application of the theory to unsymmetrical apparatus. For instance, suppose the experimental set-up consists of a bar a meter or so in length on which the subject is to set a pointer to predict the occurrence of a spot of light. It seems unreasonable to suppose that the conditioning effect of a reinforcement near the end points of the bar will be smeared symmetrically to the left and to the right. For such a situation the mean of the smearing distribution (of a sampled stimulus) may not be at the point of reinforcement even though conditioning is effective. On the other hand, it seems psychologically sound to assume that the mode of the smearing distribution will be at the point of reinforcement—granted the effectiveness of conditioning. In the present formulation of the theory it is essential to have the one free parameter of the smearing distribution closely tied to the points of reinforcement, for when conditioning is effective, which occurs with probability θ, this parameter assumes the value of the point of reinforcement (Axiom C2). This corresponds to the assumption in the finite response case that with probability θ sampled stimuli become conditioned or connected to the reinforced response.

The remaining conditioning axioms (C3, C4, C5) have almost exactly the form which is also appropriate for the finite case. The same is true of the two sampling axioms. In contrast, the first response axiom, R1, has a much simpler form in the finite case: with probability 1 the response is made to which the sampled stimulus is conditioned. Axiom R1 generalizes this assumption in the obvious manner in terms of the smearing distribution of the sampled stimulus.

The three axioms C5, S2, and R2 are what have been termed in the literature *independence-of-path* assumptions. Only R2 is new here; the other two are also needed in the finite case. These three axioms are crucial in proving that for simple reinforcement schedules the sequence of random variables which take as values the modes of the smearing distributions of the stimuli constitutes a continuous-state Markov process.

We next introduce some notation. In particular, we need notation for five random variables, their values, and their distributions, as well as a notation for their joint distribution. Three of these random variables take values in the interval $[a, b]$, the continuum of possible responses and reinforcements fixed throughout the paper. Thus we have for trial n:

(i) the *response* random variable $\boldsymbol{X}_n$, with values x_n or simply x, distribution R_n, and density r_n;

(ii) the *reinforcement* random variable Y_n, with values y_n or y, distribution F_n, and density f_n;

(iii) the *smearing-parameter* random variable $\boldsymbol{Z}_{s,n}$ of stimulus s, with values $z_{s,n}$ or z_s, distribution $G_{s,n}$, and density $g_{s,n}$. As indicated already, z_s is the mode of the smearing distribution of stimulus s. The random variable $\boldsymbol{Z}_n$, without the subscript s, shall take as values finite vectors $z = (z_{s_1}, \cdots, z_{s_N})$ relative to the ordering $(s_1, \cdots, s_N)$ of the set S of stimuli.

We also need for occasional use:

(iv) the *sampling* random variable $\boldsymbol{S}_n$, with values s_n or s for the sampled stimulus, and discrete density σ_n (it is always assumed that the set S of stimuli is finite);

(v) the *effectiveness-of-conditioning* random variable $\boldsymbol{D}_n$, with value 1 for effective and 0 for non-effective, and probability θ of value 1, following Axiom C2. I use $\delta_{t,n}$ for values of D_n. Thus $\delta_{t,n}$ is always either 1 or 0.

I use J_n for the joint distribution of any finite sequence of these random variables the last of which occurs on trial n, and j_n for the corresponding density. For occasional reference to points in the underlying sample space, ξ is used. Finally, the notation $K_s(x_n; z_n)$ for the smearing distribution of stimulus s was introduced earlier.

In terms of the five random variables introduced, the postulated sequence of events on any trial, which was described informally before, may be symbolized as follows:

$$\boldsymbol{Z}_n \to \boldsymbol{S}_n \to \boldsymbol{X}_n \to \boldsymbol{Y}_n \to \boldsymbol{D}_n \to \boldsymbol{Z}_{n+1} \ .$$

Note that the value of the random variable $\boldsymbol{Z}_n$ represents the conditioning of each stimulus at the beginning of trial n, for in the present continuous theory conditioning is in terms of a one-parameter family of smearing distributions.

It will also be useful to give a more precise formulation of the response axioms, R1 and R2, in terms of the notation just introduced. It is intended that R1 should simply make the following assertion:

$$P(a_1 \leqq \boldsymbol{X}_n \leqq a_2 \mid \boldsymbol{S}_n = s, \boldsymbol{Z}_{s,n} = z) = \int_{a_1}^{a_2} j_n(x \mid s, z)\, dx = K_s(a_2; z) - K_s(a_1; z) .$$

Axiom R2 states an independence-of-path assumption. Let w_{n-1} be any sequence of outcomes of the random variables defined up to trial $n - 1$. Then R2 asserts:

$$\int_{a_1}^{a_2} j_n(x \mid s_n, z_{s,n}, w_{n-1})\, dx = \int_{a_1}^{a_2} j_n(x \mid s_n, z_{s,n})\, dx = K_s(a_2; z) - K_s(a_1; z) .$$

The following obvious relations for the response density r_n will also be helpful later. First, we have that

$$r_n(x) = j_n(x) ,$$

i.e., r_n is just the marginal density obtained from the joint distribution j_n. Second, we have "expansions" like

$$r_n(x) = \int_a^b j_n(x, z_{s,n})\, dz_{s,n} ,$$

$$r_n(x) = \int_a^b \int_a^b \int_a^b j_n(x, z_{s,n}, y_{n-1} x_{n-1})\, dz_{s,n}\, dy_{n-1}\, dx_{n-1} .$$

3. General Theorems

This section contains five general theorems, most of which correspond to theorems that have proved useful in experimental work with the finite case. It is assumed that the reinforcement distribution F_n, which is selected by the experimenter, is always continuous and piecewise differentiable in all variables. Under these assumptions and those of Axiom C1 on the smearing distributions, no questions of integrability arise. Proofs of the first theorems are rather explicit in order to indicate the role of the axioms.

THEOREM 1. (*General Response Theorem.*)

$$r_n(x) = \sum_{s \in S} \sigma_n(s) \int_a^b k_s(x; z_s) g_{s,n}(z_s)\, dz_s . \tag{1}$$

PROOF. Mainly by virtue of Axiom S1, which asserts that exactly one stimulus is sampled on each trial,

$$\begin{aligned} r_n(x) &= \sum_S \int_a^b j_n(x, s, z_s)\, dz_s \\ &= \sum_S \int_a^b j_n(x \mid s, z_s) j_n(s \mid z_s) j_n(z_s)\, dz_s . \end{aligned} \tag{2}$$

In view of Axioms C1 and R1,

$$j_n(x \mid s, z_s) = k_s(x; z_s) ; \tag{3}$$

from Axiom S2, the independence-of-path assumption on sampling,

$$j_n(s \mid z_s) = \sigma_n(s) ; \tag{4}$$

and on the basis of the notation introduced in the last section,

(5) $$j_n(z_s) = g_{s,n}(z_s) \ .$$

The theorem follows immediately from (2)–(5). Q.E.D.

The next theorem asserts the Markov property, which is essential for further deductive developments of the theory. It is a straightforward matter to generalize this theorem to more complicated reinforcement distributions which depend on the actual responses or reinforcements on several preceding trials; the generality of the present theorem is sufficient for our purposes here.

THEOREM 2. (*Markov Theorem.*) *If the reinforcement distribution $F(y)$ on trial n is independent of n and depends only on the immediately preceding response on trial n, then the sequence of random variables $\langle \boldsymbol{Z}_1, \boldsymbol{Z}_2, \cdots, \boldsymbol{Z}_n, \cdots \rangle$ is a continuous-state Markov process.*

PROOF. By direct probability considerations for $t_1, \cdots, t_m > 1$,

(6) $$\begin{aligned} j_n(z_n \mid z_{n-1}, z_{n-t_1}, \cdots, z_{n-t_m}) = \\ \sum_i \int_a^b \int_a^b \sum_{s \in S} j_n(z_n \mid \delta_{i,n-1}, y_{n-1}, x_{n-1}, s_{n-1}, z_{n-1}, z_{n-t_1}, \cdots, z_{n-t_m}) \\ \cdot j_{n-1}(\delta_{i,n-1} \mid y_{n-1}, x_{n-1}, s_{n-1}, z_{n-1}, z_{n-t_1}, \cdots, z_{n-t_m}) \\ \cdot j_{n-1}(y_{n-1} \mid x_{n-1}, s_{n-1}, z_{n-1}, z_{n-t_1}, \cdots, z_{n-t_m}) \\ \cdot j_{n-1}(x_{n-1} \mid s_{n-1}, z_{n-1}, z_{n-t_1}, \cdots, z_{n-t_m}) \\ \cdot j_{n-1}(s_{n-1} \mid z_{n-1}, z_{n-t_1}, \cdots, z_{n-t_m})\, dy_{n-1} dx_{n-1} \ . \end{aligned}$$

Now by Axiom C2, if $\delta_{i,n-1} = 1$, then

$$j_n(z_n \mid \delta_{i,n-1}, y_{n-1}, x_{n-1}, s_{n-1}, z_{n-1}, z_{n-t_1}, \cdots, z_{n-t_m}) = 1 \ ,$$

provided the vector $z_n = y_{n-1}$ in its coordinate for stimulus s; otherwise $j_n(z_n \mid \cdots) = 0$. And if $\delta_{i,n-1} = 0$, then $j_n(z_n \mid \cdots) = 1$ if $z_n = z_{n-1}$; otherwise $j_n(z_n \mid \cdots) = 0$. For any of these cases, the value of $j_n(z_n \mid \cdots)$ is not affected by $z_{n-t_1}, \cdots, z_{n-t_m}$. Second, by virtue of Axiom C5,

$$j_{n-1}(\delta_{i,n-1} \mid y_{n-1}, x_{n-1}, s_{n-1}, z_{n-1}, z_{n-t_1}, \cdots, z_{n-t_m}) = j_{n-1}(\delta_{i,n-1}) \ .$$

Third, on the basis of the hypothesis of the theorem,

$$j_{n-1}(y_{n-1} \mid x_{n-1}, s_{n-1}, z_{n-1}, z_{n-t_1}, \cdots, z_{n-t_m}) = f(y_{n-1} \mid x_{n-1}) \ .$$

Fourth, in view of Axioms R1 and R2,

$$j_{n-1}(x_{n-1} \mid s_{n-1}, z_{n-1}, z_{n-t_1}, \cdots, z_{n-t_m}) = j_{n-1}(x_{n-1} \mid s_{n-1}, z_{n-1}) \ .$$

Finally, in view of Axiom S2,

$$j_{n-1}(s_{n-1} \mid z_{n-1}, z_{n-t_1}, \cdots, z_{n-t_m}) = \sigma_{n-1}(s_{n-1}) \ .$$

When all these results of applying the independence-of-path assumptions are substituted in (6), and the summations and integrations are performed on the result, we have

$$j_n(z_n \mid z_{n-1}, z_{n-t_1}, \cdots, z_{n-t_m}) = j_n(z_n \mid z_{n-1}) ,$$

the desired result. Q.E.D.

Some readers may feel that the above theorem could have been assumed as an axiom, but this is to misunderstand the character of the theorem in the context of the general stimulus-sampling theory formulated by the axioms. The axioms on which this theorem is based are of a general nature and are concerned with fundamental aspects of the postulated psychological process of learning. In contrast, the theorem is relatively restricted, dealing as it does with only a small class of the possible schedules of reinforcement.

We turn now to some recursion theorems for various quantities; of particular interest is the one for response probabilities. It is possible to state and prove these theorems under the general assumption of N stimuli in the set S. However, both computations and notation become rather cumbersome, so that at this stage of development of the theory it is a reasonable simplification to impose the following

RESTRICTIVE HYPOTHESIS: *There is exactly one stimulus element in S.*

Probabilities enter the theory for a continuum of responses in so many different ways that it is difficult to distinguish empirically between models with different numbers of stimuli when the stimulation is constant. And in the case of discrimination experiments, each stimulating situation may be treated as a single stimulus, with the result that on any trial there is exactly one stimulus available for sampling, although the set S may contain more than one element. As a matter of fact, this restrictive hypothesis of a single stimulus is already a practical necessity for complicated reinforcement situations in the finite case (see, for instance, [1]).

We begin with a recursion for the distribution g_n of the smearing parameter z of the single stimulus. (On the assumption of a single stimulus we drop the subscript s.)

THEOREM 3.

$$g_{n+1}(z) = (1 - \theta)g_n(z) + \theta f_n(z) . \tag{7}$$

PROOF. By Axiom C2, if conditioning is effective, then $z_{n+1} = y_n$, and thus the distribution of z_{n+1} is that of y_n, which is f_n. On the other hand, if conditioning is not effective, then $z_{n+1} = z_n$, and thus the distribution of z_{n+1} is simply g_n. By Axiom C2 the probability of the first alternative is θ, and that of the second $1 - \theta$, which yields the theorem. Q.E.D.

In the familiar notation of the finite case, where $A_{i,n}$ is response i on trial n and $E_{j,n}$ is reinforcing event j on trial n, (7) corresponds to:

$$P(A_{i,n+1}) = (1 - \theta)P(A_{i,n}) + \theta P(E_{i,n}) . \tag{8}$$

For the response density r_n we have

THEOREM 4.

$$r_{n+1}(x) = (1-\theta)r_n(x) + \theta\int_a^b k(x;y)f_n(y)\,dy\ . \tag{9}$$

PROOF. We have at once from Theorem 1

$$r_{n+1}(x) = \int_a^b k(x;z)g_{n+1}(z)\,dz\ .$$

Applying Theorem 3 to the right-hand side, we have

$$\begin{aligned} r_{n+1}(x) &= \int_a^b k(x;z)[(1-\theta)g_n(z) + \theta f_n(z)]\,dz \\ &= (1-\theta)\int_a^b k(x;z)g_n(z) + \theta\int_a^b k(x;z)f_n(z)\,dz \\ &= (1-\theta)r_n(z) + \theta\int_a^b k(x;y)f_n(y)\,dy\ , \end{aligned}$$

where the variable of integration is changed in the second integral on the right. Q.E.D.

Robert R. Bush suggested that it is of interest to see what happens when the interval $[a, b]$ is cut into a finite number of parts and the resulting finite response case is studied. For simplicity, we may divide the interval into exactly two parts. Let $a < c < b$, and call $X_{1,n}$ a response on trial n in the interval $[a, c]$, and $X_{2,n}$ a response on trial n in $[c, b]$. Clearly

$$\begin{aligned} P(X_{1,n}) &= R_n(c) - R_n(a) = R_n(c)\ , \\ P(X_{2,n}) &= R_n(b) - R_n(c) = 1 - R_n(c)\ . \end{aligned}$$

And by integrating (9) of Theorem 4, we have at once

THEOREM 5.

$$\begin{aligned} P(X_{1,n+1}) &= (1-\theta)P(X_{1,n}) + \theta\int_a^c\int_a^b k(x;y)f_n(y)\,dx\,dy\ , \\ P(X_{2,n+2}) &= (1-\theta)P(X_{2,n}) + \theta\int_c^b\int_a^b k(x;y)f_n(y)\,dx\,dy\ . \end{aligned} \tag{10}$$

The recursions for $X_{1,n}$ and $X_{2,n}$ may be regarded as a generalization of (8) for the finite case when a continuous smearing of the effects of reinforcement is postulated. By further specialization, it is possible to get an exact analog of (8). Let us suppose that there are only two points of reinforcement, one the midpoint y_1 of the interval $[a, c]$, and the other the midpoint y_2 of the interval $[c, b]$. Suppose, moreover, that the smearing densities around these two points of reinforcement are strictly positive only in the subinterval $[a, c]$ or $[c, b]$ as the case may be. Define then

$$Y_{1,n} = \int_a^c k(x;y_1)\,dx, \qquad Y_{2,n} = \int_c^b k(x;y_2)\,dx\ ,$$

and under these suppositions (10) becomes

$$P(X_{i,n+1}) = (1-\theta)P(X_{i,n}) + \theta P(Y_{i,n}) ,$$

an exact analog of (8). (Naturally, weaker suppositions will also yield such an analog, but the present example is illustrative of one method for obtaining the finite case from the continuous one.)

The suppositions just made to yield (8) may also be used to yield the standard theory of the finite case at a deeper level, for (8) is only a recursion in the mean probabilities of responses and in itself does not justify derivation of any sequential statistics like the probability of two successive A_1 responses. However, these matters will not be pursued further here.

In connection with this comparison of models, it may also be remarked that the response density recursion (9) of Theorem 4 is exactly the same as that obtained in [3] for the continuous-response linear model. Consequently, the results in [3] for various kinds of contingent reinforcement (and *a fortiori* non-contingent reinforcement) follow at once in the present theory.

4. Non-contingent Reinforcement

For non-contingent reinforcement schedules—that is, those for which the distribution $F(y)$ is independent of n and the past—we first use the response density recursion (9) to prove some simple, useful results which do not explicitly involve the smearing distribution of the single stimulus element and which also hold in the linear model but were not stated in [3]. There is, however, one necessary preliminary concerning derivation of the asymptotic response distribution in the stimulus-sampling theory.

THEOREM 6. *In the non-contingent case*

$$r(x) = \lim_{n\to\infty} r_n(x) = \int_a^b k(x;y)f(y)\,dy . \tag{11}$$

PROOF. Because in the non-contingent case $f_n(y) = f(y)$, we have at once from Theorem 3

$$g(z) = \lim_{n\to\infty} g_n(z) = f(z) . \tag{12}$$

The theorem immediately follows from (12) and Theorem 1. Q.E.D.

We now use (11) to establish the following recursions. In the statement of the theorem $\mathscr{E}(X_n)$ is the expectation of the response random variable X_n; $\mu_r(X_n)$ is its rth raw moment; $\sigma^2(X_n)$ is its variance; and X is the random variable with density r.

THEOREM 7.

$$r_{n+1}(x) = (1-\theta)r_n(x) + \theta r(x) , \tag{13}$$

$$\mathscr{E}(X_{n+1}) = (1-\theta)\mathscr{E}(X_n) + \theta\mathscr{E}(X) , \tag{14}$$

$$\mu_r(X_{n+1}) = (1-\theta)\mu_r(X_n) + \theta\mu_r(X) , \tag{15}$$

$$\sigma^2(X_{n+1}) = (1-\theta)\sigma^2(X_n) + \theta\sigma^2(X) + \theta(1-\theta)[\mathscr{E}(X_n) - \mathscr{E}(X)]^2 . \tag{16}$$

PROOF. Because $f_n(y) = f(y)$ in the non-contingent case, (13) follows at once from (9) and (11), i.e., from Theorems 4 and 6. Multiplying both sides of (13) by x^r and integrating over the interval $[a, b]$, we obtain (15), of which (14) is a special case. As for (16), we infer it from the following:

$$\begin{aligned}\sigma^2(X_{n+1}) &= \mu_2(X_{n+1}) - \mathscr{E}(X_{n+1})^2 \\ &= (1-\theta)\mu_2(X_2) + \theta\mu_2(X) - (1-\theta)^2\mathscr{E}(X_n) \\ &\qquad - 2\theta(1-\theta)\mathscr{E}(X_n)\mathscr{E}(X) - \theta^2\mathscr{E}(X)^2 \\ &= (1-\theta)[\mu_2(X_n) - \mathscr{E}(X_n)^2] + \theta[\mu_2(X) - \mathscr{E}(X)^2] \\ &\qquad + (\theta-\theta^2)\mathscr{E}(X_n)^2 - 2(\theta-\theta^2)\mathscr{E}(X_n)\mathscr{E}(X) + (\theta-\theta^2)\mathscr{E}(X)^2 \\ &= (1-\theta)\sigma^2(X_n) + \theta\sigma^2(X) + \theta(1-\theta)[\mathscr{E}(X) - \mathscr{E}(X_n)]^2. \text{ Q.E.D.}\end{aligned}$$

Because (13)–(15) are first-order difference equations with constant coefficients we have as an immediate consequence of the theorem:

COROLLARY 7.1.

$$r_n(x) = r(x) - [r(x) - r_1(x)](1-\theta)^{n-1}, \tag{17}$$

$$\mathscr{E}(X_n) = \mathscr{E}(X) - [\mathscr{E}(X) - \mathscr{E}(X_1)](1-\theta)^{n-1}, \tag{18}$$

$$\mu_r(X_n) = \mu_r(X) - [\mu_r(X) - \mu_r(X_1)](1-\theta)^{n-1}. \tag{19}$$

Although the linear and (one-element) stimulus-sampling models both yield (13)–(19), predictions in the two models are already different for one of the simplest sequential statistics, namely, the probability of two successive responses in the same or different subintervals.

For two subintervals $[a, c]$ and $[c, b]$, we have the following theorem for the stimulus-sampling model. The result generalizes directly to any finite number of subintervals.

THEOREM 8. *For non-contingent reinforcement*

$$\begin{aligned}&\lim_{n\to\infty} P(a \leqq X_{n+1} \leqq c,\ a \leqq X_n \leqq c) \\ &\qquad = \theta R(c)^2 + (1-\theta)\int_a^c\int_a^c\int_a^b k(x; z)k(x'; z)f(z)\,dx\,dx'dz,\end{aligned} \tag{20}$$

$$\begin{aligned}&\lim_{n\to\infty} P(a \leqq X_{n+1} \leqq c,\ c \leqq X_n \leqq b) \\ &\qquad = \theta R(c)[1 - R(c)] + (1-\theta)\int_a^c\int_c^b\int_a^b k(x; z)k(x'; z)f(z)\,dx\,dx'dz,\end{aligned} \tag{21}$$

where

$$R(c) = \lim_{n\to\infty} R_n(c).$$

PROOF. We first establish (20). To begin with,

$$P(a \leqq X_{n+1} \leqq c,\ a \leqq X_n \leqq c) = \int_a^c\int_a^c j_{n+1}(x_{n+1}, x_n)\,dx_{n+1}\,dx_n.$$

Applying the axioms in the usual way to the right-hand side, we obtain

$$\int_a^c\int_a^c j_{n+1}(x_{n+1}, x_n)\, dx_{n+1}dx_n$$

$$= \int_a^c\int_a^b \sum_i \int_a^b\int_a^c\int_a^b j_{n+1}(x_{n+1}, z_{n+1}, \delta_{i,n}, y_n, x_n, z_n) \cdot dx_{n+1}\, dz_{n+1}\, dy_n\, dx_n\, dz_n$$

$$= \int_a^c\int_a^b \sum_i \int_a^b\int_a^c\int_a^b j(x_{n+1} \mid z_{n+1}) j(z_{n+1} \mid \delta_{i,n}, y_n, x_n, z_n) j(\delta_{i,n})$$

$$\cdot f(y_n) j(x_n \mid z_n) j(z_n)\, dx_{n+1}\, dz_{n+1}\, dy_n\, dx_n\, dz_n$$

$$= \int_a^c\int_a^b\int_a^c\int_a^b [k(x_{n+1}; y_n)\theta f(y_n) k(x_n; z_n) g_n(z_n)$$

$$+ k(x_{n+1}; z_n)(1-\theta) k(x_n; z_n) g_n(z_n)]\, dx_{n+1}\, dy_n\, dx_n\, dz_n \ .$$

Now

$$\lim_{n\to\infty} g_n(z) = f(z) \ ,$$

whence at asymptote, by rearranging the right-hand side and relettering variables, we obtain

$$\lim_{n\to\infty} P(a \leqq X_{n+1} \leqq c,\ a \leqq X_n \leqq c)$$

$$= \theta\left(\int_a^c\int_a^b k(x; y) f(y)\, dx\, dy\right)\left(\int_a^c\int_a^b k(x'; z) f(z)\, dx' dz\right)$$

$$+ (1-\theta)\int_a^c\int_a^c\int_a^b k(x; z) k(x'; z) f(z)\, dx\, dx' dz \ .$$

But the first term on the right is just $\theta R(c)^2$, which when substituted in yields (20).

The argument establishing (21) proceeds along exactly the same lines, with functions of x_n now integrated over the interval $[c, b]$. Q.E.D.

For comparative purpose the corresponding results for the linear model are derived in the Appendix.

The theorem just proved may be used to develop a reasonably good method of estimating the learning parameter θ. The sequence of response random variables $<A_1, A_2, \cdots, A_n, \cdots>$, where

$$A_n = \begin{cases} 1 & \text{if response on trial } n \text{ is in interval } [a, c] \ , \\ 2 & \text{otherwise} \ , \end{cases}$$

is a chain of infinite order. If it were a first-order Markov chain, (20) and (21) could be used to obtain a maximum likelihood estimate of θ. The estimate θ^* proposed here is formally identical with the latter, but of course it is not the maximum likelihood estimate. I shall call it the *pseudo-maximum likelihood* estimate.

Let $a_1, a_2, \cdots, a_n$ represent a finite sequence of values of the response random variables $A_1, A_2, \cdots, A_n$ from trial 1 to trial n. Let s be the number of subjects. Then, granted statistical independence of the subjects, the maximum likelihood estimate of θ is the number $\hat{\theta}$ (if it exists) such that for all θ'

$$(22)\qquad \prod_{\sigma=1}^{s} f^{(\sigma)}(a_1, a_2, \cdots, a_n; \hat{\theta}) \geqq \prod_{\sigma=1}^{s} f^{(\sigma)}(a_1, a_2, \cdots, a_n; \theta') ,$$

where $f^{(\sigma)}(a_1, a_2, \cdots, a_n; \hat{\theta})$ is the probability of the sequence of responses $a_1, a_2, \cdots, a_n$ for subject σ when the learning parameter is $\hat{\theta}$.

As should be clear from preceding remarks, the pseudo-maximum likelihood estimate of θ is the number θ^* such that for all θ'

$$(23)\qquad \prod_{\sigma=1}^{s} \prod_{m=2}^{n} f^{(\sigma)}(a_m \mid a_{m-1}; \theta^*) f^{(\sigma)}(a_1; \theta^*) \geqq \prod_{\sigma=1}^{s} \prod_{m=2}^{n} f^{(\sigma)}(a_m \mid a_{m-1}; \theta') f^{(\sigma)}(a_1; \theta') .$$

To simplify notation, let $p_{ij}(\theta)$ be the probability of going from state i to state j $(i, j = 1, 2)$ with parameter θ; let n_{ij} be the number of actual transitions from state i to state j, summed over trials and subjects (the n_{ij} are tabulated from experimental data); let $p_i(\theta)$ be the probability of being in state i on trial 1; and let n_i be the number of subjects in state i on trial 1. We then want to find the θ that maximizes

$$\prod_{i,j} p_i^{n_i}(\theta) p_{ij}^{n_{ij}}(\theta) .$$

It is usually easier to work with the log of this expression, so we seek to maximize

$$(24)\qquad L^*(\theta) = \sum_i \left(n_i \log p_i(\theta) + \sum_j n_{ij} \log p_{ij}(\theta) \right) .$$

In most case $L^*(\theta)$ has a local maximum, so we can find θ^* as an appropriate solution of

$$(25)\qquad \frac{dL^*(\theta)}{d\theta} = \sum_i \left(\frac{n_i p_i'(\theta)}{p_i(\theta)} + \sum_j \frac{n_{ij} p_{ij}'(\theta)}{p_{ij}(\theta)} \right) = 0 ,$$

where p' is the derivative of p with respect to θ.

Now on the basis of (20) and (21), at asymptote we have

$$(26)\qquad p_{11}(\theta) = \theta R(c) + \frac{(1-\theta)}{R(c)} \int_a^c \int_a^c \int_a^b k(x; z) k(x'; z) f(z)\, dx\, dx' dz$$

and

$$(27)\qquad p_{21}(\theta) = \theta R(c) + \frac{(1-\theta)}{1 - R(c)} \int_a^c \int_c^b \int_a^b k(x; z) k(x'; z) f(z)\, dx\, dx' dz ,$$

and $p_i(\theta)$ is independent of θ. Also, of course, $p_{12}(\theta) = 1 - p_{11}(\theta)$, and $p_{22}(\theta) = 1 - p_{21}(\theta)$. Moreover,

$$(28)\qquad p_{11}'(\theta) = R(c) - \frac{\alpha}{R(c)} , \qquad p_{22}'(\theta) = R(c) - \frac{\beta}{1 - R(c)} ,$$

where

$$\alpha = \int_a^b K(c; z)^2 f(z)\, dz = \int_a^c \int_a^c \int_a^b k(x; z)\, k(x'; z)\, f(z)\, dx\, dx'\, dz \tag{29}$$

and

$$\beta = \int_a^c \int_c^b \int_a^b k(x; z)\, k(x'; z)\, f(z)\, dx\, dx'\, dz$$
$$= \int_a^b K(c; z)(1 - K(c; z)) f(z)\, dz = R(c) - \alpha\,. \tag{30}$$

Applying (26)-(30) to (25) and using the fact that $p_i(\theta)$ is independent of θ, we obtain:

$$\frac{dL^*(\theta)}{d\theta} = \frac{n_{11}\left(R(c) - \dfrac{\alpha}{R(c)}\right)}{\theta R(c) + \dfrac{(1-\theta)\alpha}{R(c)}} + \frac{n_{12}\left(\dfrac{\alpha}{R(c)} - R(c)\right)}{1 - \theta R(c) - \dfrac{(1-\theta)\alpha}{R(c)}}$$
$$+ \frac{n_{21}\left(R(c) - \dfrac{\beta}{1 - R(c)}\right)}{\theta R(c) + \dfrac{(1-\theta)\beta}{1 - R(c)}} + \frac{n_{22}\left(\dfrac{\beta}{1 - R(c)} - R(c)\right)}{1 - \theta R(c) - \dfrac{(1-\theta)\beta}{1 - R(c)}} = 0\,. \tag{31}$$

Solving (31), we have

THEOREM 9. *If $r_1(x) = r(x)$ for all x in $[a, b]$, then the estimate θ^* is a solution of the quadratic equation*

$$N\theta^2 + [(N - n_{11})A + (n_{11} + n_{22})B + (N - n_{22})C]\theta$$
$$+ n_{22}AB + (n_{12} + n_{21})AC + n_{11}BC = 0\,, \tag{32}$$

where

$$A = \alpha/[R(c)^2 - \alpha], \qquad B = -[R(c) - \alpha]/[R(c)^2 - \alpha]\,,$$
$$C = [1 + \alpha - 2R(c)]/[R(c)^2 - \alpha], \qquad N = \sum_{i,j} n_{ij}\,.$$

Moreover, if $R(c) = \frac{1}{2}$, then

$$\theta^* = -\frac{A(n_{12} + n_{21}) + B(n_{11} + n_{22})}{N}\,.$$

Note that the hypothesis of the theorem simply requires that we start counting trials at asymptote. The statistical properties of the estimator θ^* need investigation; it can be shown to be consistent.

I conclude the treatment of non-contingent reinforcement with two expressions dealing with important sequential properties of stimulus-sampling models. The first gives the probability of a response in the interval $[a_1, a_2]$ given that on the previous trial the reinforcing event occurred in the interval $[b_1, b_2]$.

THEOREM 10.

(33) $$P(a_1 \leqq X_{n+1} \leqq a_2 \mid b_1 \leqq Y_n \leqq b_2) = (1-\theta)[R_n(a_2) - R_n(a_1)] + \frac{\theta}{F(b_2) - F(b_1)} \int_{a_1}^{a_2}\int_{b_1}^{b_2} k(x; y) f(y)\, dx\, dy .$$

PROOF. By the usual expansion

$$P(a_1 \leqq X_{n+1} \leqq a_2 \mid b_1 \leqq Y_n \leqq b_2) = \frac{1}{F(b_2) - F(b_1)} \cdot \int_{a_1}^{a_2}\int_a^b \sum_i \int_{b_1}^{b_2}\int_a^b j_{n+1}(x_{n+1}, z_{n+1}, \delta_{i,n}, y_n, z_n)\, dx_{n+1}\, dz_{n+1}\, dy_n\, dz_n .$$

And the right-hand side is

$$\frac{1}{F(b_2) - F(b_1)} \left[(1-\theta) \int_{a_1}^{a_2}\int_{b_1}^{b_2}\int_a^b k(x; z) g_n(z) f(y)\, dx\, dy\, dz + \theta \int_{a_1}^{a_2}\int_{b_1}^{b_2}\int_a^b k(x; y) f(y) g_n(z)\, dx\, dy\, dz \right] .$$

Now in the first and second terms, respectively, we have

$$\int_{b_1}^{b_2} f(y)\, dy = F(b_2) - F(b_1) \quad \text{and} \quad \int_a^b g_n(z)\, dz = 1 .$$

Using these two results, we obtain the theorem at once. Q.E.D.

The expression to which we now turn gives the probability of a response in the interval $[a_1, a_2]$ given that on the previous trial the reinforcing event occurred in the interval $[b_1, b_2]$ and the response in the interval $[a_3, a_4]$.

THEOREM 11.

(34) $$P(a_1 \leqq X_{n+1} \leqq a_2 \mid b_1 \leqq Y_n \leqq b_2,\ a_3 \leqq X_n \leqq a_4) = \frac{(1-\theta)}{R_n(a_4) - R_n(a_3)} \int_{a_1}^{a_2}\int_{a_3}^{a_4}\int_a^b k(x; z) k(x'; z) g_n(z)\, dx\, dx'\, dz + \frac{\theta}{F(b_2) - F(b_1)} \int_{a_1}^{a_2}\int_{b_1}^{b_2} k(x; y) f(y)\, dx\, dy .$$

PROOF. It is first useful to observe that for non-contingent reinforcement

$$\begin{aligned} P(b_1 \leqq Y_n \leqq b_2,\ a_3 \leqq X_n \leqq a_4) &= P(b_1 \leqq Y_n \leqq b_2 \mid a_3 \leqq X_n \leqq a_4) P(a_3 \leqq X_n \leqq a_4) \\ &= P(b_1 \leqq Y_n \leqq b_2) P(a_3 \leqq X_n \leqq a_4) \\ &= [F(b_2) - F(b_1)][R_n(a_4) - R_n(a_3)] . \end{aligned}$$

Applying the usual expansion to the left-hand quantity in (34), we obtain

$$\frac{1}{[F(b_2) - F(b_1)][R_n(a_4) - R_n(a_3)]} \cdot \int_{a_1}^{a_2}\int_a^b \sum_i \int_{b_1}^{b_2}\int_{a_1}^{a_2}\int_a^b j_{n+1}(x_{n+1}, z_{n+1}, \delta_{i,n}, y_n, x_n, z_n)\, dx_{n+1}\, dz_{n+1}\, dy_n\, dx_n\, dz_n ,$$

from which, using particularly Axioms C2 and C5, we have

$$\frac{1}{[F(b_2) - F(b_1)][R_n(a_4) - R_n(a_3)]} \cdot \left[(1-\theta)\int_{a_1}^{a_2}\int_{b_1}^{b_2}\int_{a_3}^{a_4}\int_{a}^{b} k(x;z)\,k(x';z)\,g_n(z)\,f(y)\,dx\,dy\,dx'\,dz \right.$$
$$\left. + \theta\int_{a_1}^{a_2}\int_{b_1}^{b_2}\int_{a_3}^{a_4}\int_{a}^{b} k(x;y)\,f(y)\,k(x';z)\,g_n(z)\,dx\,dy\,dx'\,dz\right].$$

Now in the first term of this last expression we may integrate out the function $f(y)$ to obtain $F(b_2) - F(b_1)$, which cancels the corresponding quantity in the denominator. Similarly, in the second term we may integrate out $k(x'; z)\,g_n(z)$ to obtain $R_n(a_4) - R_n(a_3)$, which for this term cancels the corresponding quantity in the denominator. Putting these results together, we have exactly the theorem. Q.E.D.

It may be noticed that by applying Corollary 7.1 more explicit results are easily obtained from both Theorems 10 and 11.

5. Simple Discrimination

It is of some interest to sketch how the present theory may be applied to simple discrimination situations in which on each trial exactly one stimulus s_i is presented, and associated with each s_i is a reinforcement distribution f^i. (Readers who do not like the idea of exactly one stimulus being presented may think of each s_i as being a particular *pattern* of stimuli.) Let the probability of presentation of s_i on any trial be ω_i, with

$$\sum_{i=1}^{N} \omega_i = 1\,, \qquad \omega_i \neq 0$$

for $i = 1, \cdots, N$, and ω_i independent of trial number and any behavior on preceding trials.

The tree of the Markov process in the states (z^1, z^2) for $N = 2$ and $\omega_i = \frac{1}{2}$ is given in Figure 1.

Corresponding to Theorem 1, we have by the same sort of proof for arbitrary N

$$r_n(x) = \sum_{i=1}^{N} \omega_i \int_a^b k_{s_i}(x; z^i)\, g_{s_i,n}(z^i)\, dz^i\,. \tag{35}$$

Corresponding to Theorem 3, we have

$$g_{n+1}(z^i) = (1-\theta)\, g_n(z^i \mid S_n = s_i) + \theta f^i(z^i)\,; \tag{36}$$

and by virtue of Axiom C4 for $i \neq j$ and $S_n = s_j$,

$$g_{n+1}(z^i) = g_n(z_i)\,, \tag{37}$$

whence it easily follows that

$$\lim_{n\to\infty} g_n(z^i) = f^i(z^i)\,. \tag{38}$$

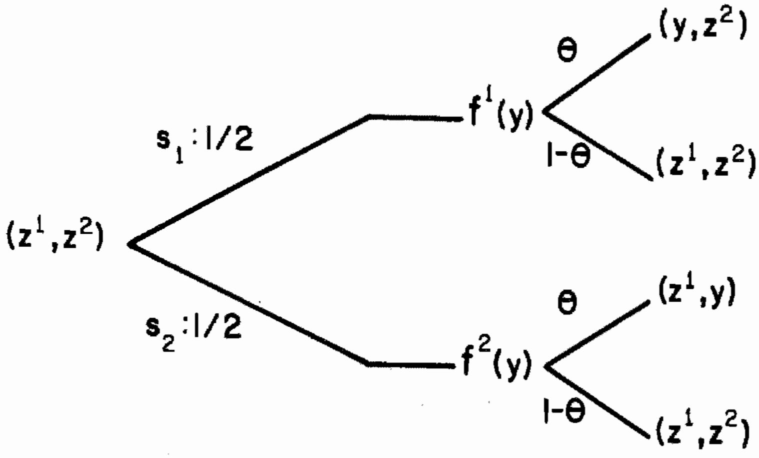

Figure 1

We then have also that

(39) $$\lim_{n\to\infty} P(a_1 \leqq \boldsymbol{X}_n \leqq a_2 \mid \boldsymbol{S}_n = s_i) = \int_{a_1}^{a_2}\int_a^b k_{s_i}(x; y) f^i(y)\, dx\, dy \ .$$

The results (35)-(39) and some other related ones that are easily obtained, although simple in character, permit application of the theory developed in this paper to simple discrimination experiments with a continuum of responses. On the other hand, it is obvious that the present theory must be modified and extended in fundamental ways to deal with discrimination experiments that have a continuum of stimuli as well as responses.

APPENDIX[1]

Our purpose is to derive for the linear model of [3] the analogs of (20) and (21). A brief description of the linear model will make the present discussion nearly self-contained. An experiment may be represented by a sequence $(\boldsymbol{X}_1, \boldsymbol{Y}_1, \boldsymbol{X}_2, \boldsymbol{Y}_2, \cdots, \boldsymbol{X}_n, \boldsymbol{Y}_n, \cdots)$ of response and reinforcement random variables. The theory is formulated for the probability of a response on trial $n+1$ given the entire preceding sequence of responses and reinforcements. For this sequence we use the notation s_n (not to be confused with the notation for the value of the sampling random variable in the main body of the paper). Aside from continuity and piecewise-differentiability assumptions, the single axiom of the linear model is

(40) $$J_{n+1}(x \mid y_n, x_n, s_{n-1}) = (1-\theta)J_n(x \mid s_{n-1}) + \theta K(x; y_n) \ ,$$

where J_n is the joint distribution and K is the smearing distribution.

[1] I am indebted to Raymond W. Frankmann for useful comments on the subject of this Appendix.

We first need to define the cross-moments

$$(41)\qquad W(a_1, a_2, a_3, a_4, n) = \int_{a_1}^{a_2}\int_{a_3}^{a_4}\int_{s_{n-1}} j_n(x \mid s_{n-1})\, j_n(x' \mid s_{n-1})\, j(s_{n-1})\, dx\, dx'\, ds_{n-1}\,,$$

where the subscript s_{n-1} on the third integration sign indicates integration over the $2(n-1)$-Cartesian product of the interval $[a, b]$ for the sequence s_{n-1}. The cross-moments defined by (41) generalize the moments $W^2_{a_1, a_2, n}$ of [3].

Assuming henceforth *non-contingent reinforcement*, it follows by simple extension of some results in [3] that

$$(42)\qquad \lim_{n\to\infty} P(a_1 \leqq X_{n+1} \leqq a_2,\ a_3 \leqq X_n \leqq a_4)$$
$$= (1-\theta) \lim_{n\to\infty} W(a_1, a_2, a_3, a_4, n) + \theta[R(a_2) - R(a_1)][R(a_4) - R(a_3)]\,.$$

To obtain an explicit answer we must compute the limit on the right, which we now proceed to do.

By virtue of the definition of s_{n-1}, the right-hand side of (41) may be rewritten, and we have

$$(43)\qquad W(a_1, a_2, a_3, a_4, n)$$
$$= \int_{a_1}^{a_2}\int_{a_3}^{a_4}\int_a^b\int_a^b\int_{s_{n-2}} j_n(x \mid y_{n-1}, x_{n-1}, s_{n-2})\, j_n(x' \mid y_{n-1}, x_{n-1}, s_{n-2})$$
$$\cdot\, j(y_{n-1}, x_{n-1}, s_{n-2})\, dx\, dx'\, dy_{n-1}\, dx_{n-1}\, ds_{n-2}\,.$$

Applying the axiom (40) to the right-hand side of (43) and simplifying, we obtain

$$(44)\qquad W(a_1, a_2, a_3, a_4, n)$$
$$= (1-\theta)^2 \int_{a_1}^{a_2}\int_{a_3}^{a_4}\int_{s_{n-2}} j_{n-1}(x \mid s_{n-2})\, j_{n-1}(x' \mid s_{n-2})\, j(s_{n-2})\, dx\, dx'\, ds_{n-2}$$
$$+ 2\theta(1-\theta) \int_{a_1}^{a_2}\int_{a_3}^{a_4}\int_a^b\int_{s_{n-2}} j_{n-1}(x \mid s_{n-2})\, j(s_{n-2})$$
$$\cdot\, k(x'; y_{n-1}) f(y_{n-1})\, dx\, dx'\, dy_{n-1}\, ds_{n-2}$$
$$+ \theta^2 \int_{a_1}^{a_2}\int_{a_3}^{a_4}\int_a^b k(x, y_{n-1})\, k(x', y_{n-1}) f(y_{n-1})\, dx\, dx'\, dy_{n-1}\,.$$

Now the first term on the right of (44) is simply $(1-\theta)^2 W(a_1, a_2, a_3, a_4, n-1)$, the second term is

$$2\theta(1-\theta)[R_{n-1}(a_2) - R_{n-1}(a_1)][R(a_4) - R(a_3)]\,,$$

and the integral of the third term is a direct generalization of β as defined by (30). Moreover, it is independent of n; and we may thus define, for ease of notation,

$$(45)\qquad \gamma(a_1, a_2, a_3, a_4) = \int_{a_1}^{a_2}\int_{a_3}^{a_4}\int_a^b k(x, y)\, k(x'; y) f(y)\, dx\, dx'\, dy\,.$$

In these terms, (44) becomes:

(46) $$W(a_1, a_2, a_3, a_4, n) = (1-\theta)^2 W(a_1, a_2, a_3, a_4, n-1) + 2\theta(1-\theta)[R_{n-1}(a_2) - R_{n-1}(a_1)][R(a_4) - R(a_3)] + \theta^2 \gamma(a_1, a_2, a_3, a_4)\,.$$

It then easily follows from (46) that

(47) $$\lim_{n\to\infty} W(a_1, a_2, a_3, a_4, n) = W(a_1, a_2, a_3, a_4) = \frac{2(1-\theta)[R(a_2) - R(a_1)][R(a_4) - R(a_3)] + \theta\gamma(a_1, a_2, a_3, a_4)}{2-\theta}\,.$$

Combining (42) and (47), we then have the following theorem.

THEOREM. *In the linear model*

(48) $$\lim_{n\to\infty} P(a_1 \leqq X_{n+1} \leqq a_2,\ a_3 \leqq X_n \leqq a_4) = \theta[R(a_2) - R(a_1)][R(a_4) - R(a_3)] + (1-\theta)\left[\frac{2(1-\theta)[R(a_2) - R(a_1)][R(a_4) - R(a_3)] + \theta\gamma(a_1, a_2, a_3, a_4)}{2-\theta}\right].$$

To obtain the direct analog of (20), (48) specializes to:

$$\lim_{n\to\infty} P(a \leqq X_{n+1} \leqq c,\ a \leqq X_n \leqq c) = \theta R(c)^2 + (1-\theta)\left[\frac{2(1-\theta)R(c)^2 + \theta\alpha}{2-\theta}\right],$$

where α is defined by (29). The analog of (21) may be obtained in like fashion.

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