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The Case for Information-oriented (Basic) Research in Mathematics Education

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THE marvelously clear and definite structure that is characteristic of most parts of modern mathematics can be misleading when problems of mathematical instruction are considered. The very clarity of the structure of mathematics itself can lead to the mistaken view that nothing beyond this structure need be considered in analyzing and deciding how mathematics should be taught.

Yet anybody who has taught mathematics knows how far from the truth this claim is. It is not a straightforward or simple matter for the average student to learn mathematics! And there is no doubt that the ordinary student finds that he has to think harder in learning mathematics than in learning just about any other subject in the curriculum.

The case for basic research in mathematics education can be stated quite simply in terms of these well-known difficulties of students. It is the ultimate objective of basic research in mathematics education to understand how students learn mathematics, and to use this understanding to outline more effective ways of organizing the curriculum. It is probably also agreed, on all sides, that we are still very far from realizing this objective. Without question, we do not yet understand in any reasonable degree of scientific detail what goes on when a student learns a piece of mathematics, whether the mathematics in question be first-grade arithmetic, undergraduate calculus, or graduate-school algebraic topology.

In this brief article I want to survey some of the more important reasons for having a vigorous program in basic research in mathematics education.

Defects of Intuition

Many teachers, who would admit that the logical structure of mathematics alone is not sufficient to determine the mathematics curriculum and how it is to be presented to students, would still maintain that the remaining gaps can be closed by appropriate use of intuition.

The first puzzling thing about this claim for intuition is that most of us have only a vague idea of what another person means when he talks about knowing something by intuition. What is intuition? We all recognize the role of experience in the training of teachers. As a rule, the teacher who has taught several years is able to do a better job than the beginner. Intuition is involved—intuition as the acquisition of knowledge and information in an inexplicit and nonformalized way on the basis of teaching experience. No one faced with the complex problems of teaching mathematics or any other part of the curriculum would want to belittle the importance of experience and practice in the training of good teachers.

Yet many examples exist in the mathematics curriculum to show that it is not sufficient to leave the curriculum to the intuition of curriculum writers and the experience of teachers. The extensive research by Brownell and others on methods of subtraction has made everyone dealing with the curriculum in arithmetic sensitive to the analysis of the actual steps that must be taught children in learning the subtraction algorithm. Another example is the evidence that in the learning of a sequence of mathematical concepts, the important problem is often to minimize negative transfer rather than to facilitate positive transfer. The existence of negative transfer in passing from one concept to another is the sort of thing that is noticed by the very good teacher; it is also the kind of phenomenon that needs to be pinned down, in terms of research, and made part of the objective evidence presented to all teachers in telling them about learning difficulties. Another example that goes contrary to the formal structure of our standard teaching of geometry is found in the clear results concerning children's perceptions of rotations and stretches of standard geometrical figures in the plane. Although Euclidean geometry uses the fundamental notion of congruence that is invariant under rotations of figures, but not under stretches in their size, at the perceptual level this notion of congruence is more difficult for young children than perceiving the relation of similarity between figures that have the same orientation and shape but different sizes. Because teachers have themselves been taught Euclidean geometry and are familiar with the concept of congruence, it is all too easy for them to infer that this is the more natural concept for children. Without supporting research, it would be difficult to convince many teachers of the true state of affairs.

Defects of Sheer Empiricism

It is also important to emphasize, in discussing the role of basic research in mathematics education, that simple applied empirical research will not answer all the many questions that confront us. For example, if we hope to determine by experimental research the optimum sequence of topics in the first two grades of elementary school (or, with equal pertinence, in the first two years of university mathematics), it is easy enough to show for either of these cases that the mathematical constraints that are placed on the possible sequences of topics are not sufficient to reduce the number of *possible* sequences of concepts to a manageable number of experiments. The number would be greater than all persons now working in mathematics education could perform in the next ten or fifteen years, even if they devoted themselves wholly to this question. The sort of mathematical constraint I have in mind is that the introduction of multiplication would, from a mathematical standpoint, have to be preceded by the introduction of addition, if multiplication is initially to be talked about in terms of repeated addition. On the other hand, there is no real reason why we could not experiment with the introduction of subtraction before addition.

Examples of a more practical nature center around questions of the following sort. Should addition and subtraction be introduced simultaneously? If not, should addition be carried to sums not greater than five, not greater than six, not greater than seven, etc., before subtraction (or at least the notation for subtraction) is introduced? Such purely empirical questions are endless in number, and I emphasize once again, there is no purely mathematical answer to them. Because there is no purely mathematical answer, the importance of a psychological theory of mathematics-learning is crucial, in order ultimately to provide appropriate answers to problems of curriculum organization.

Another way of putting the matter is that purely empirical research lacks conceptual power, because the absence of any theory prohibits us from making extensive generalizations to other situations and broader classes of problems.

From this standpoint, I would emphasize that the demands for a psychological theory of mathematics-learning, and thus for theoretical basic research as well as empirical basic research, are practical demands. Without such theory it is impossible for us to answer in any scientific way many substantive questions of curriculum organization. The vast literature on readiness, drills, practice, and overlearning in arithmetic and other subjects has made all of us aware of the complex and subtle nature of the empirical problems. Anyone who thinks that he can answer these problems either by intuition or by any simple experimental program, without facing the theoretical problems of weaving into one coherent theoretical pattern the many kinds of results already obtained, is surely daydreaming.

In this discussion of empirical problems I have emphasized the kind of questions that have arisen in elementary-school mathematics. The reason for this is simply that a greater body of research already exists in this area. The problems of mathematics-learning at the university level are certainly more complex and difficult, and may demand even more of an effort in basic research in order to begin to understand them.

Analysis of Learning Difficulties

Given a particular organization of the curriculum in terms of the concepts to be taught and the sequence in which these concepts will be presented, it is still a major task of basic research to analyze and provide a theory for the kind of learning difficulties students encounter as they progress through this curriculum. It is again important to emphasize that the learning difficulties students encounter cannot be predicted by a nonpsychological mathematical analysis of the mathematical content of the curriculum itself—at least no one has proposed such a theory, and there are good reasons for thinking that no such theory shall be proposed.

It is not a part of arithmetic proper or of geometry proper to make psychological predictions about the difficulties students will have with the different concepts in these disciplines. It is the task of a psychological theory of mathematics-learning to predict and to offer an analysis of the kinds of difficulties that are encountered. The success of mathematics teaching depends upon understanding and providing successful practical remedies for the difficulties that students do encounter. In our increasingly technological age it is of greater importance than ever before that we, as educators, recognize the need for clear analysis of students' learning difficulties and the pressing need to develop theories that adequately deal with these difficulties. I have tried to emphasize in this brief discussion that neither intuition nor sheer empiricism is able to provide adequate answers to our problems. I have rested the case for basic research on the overwhelming practical importance of the solutions one hopes to find. I would like to conclude with some remarks in a somewhat different direction.

Psychology of Learning and the Nature of Mathematics

It is my own conjecture that as we are able to dig deeper into the development of an adequate psychological theory of mathematics-learning, the results will have an impact on our conception of the nature of mathematics itself.

It is not possible here to defend this conjecture in a detailed way, but there is reason to think that concentration on mathematical thinking and the difficulties students have in learning to think mathematically will lead to a new conception of *invariance,* a conception that goes beyond that now encountered in the various parts of mathematics. Historically, the standard philosophies of mathematics have emphasized differing attitudes toward the nature of mathematical objects, but it is perfectly obvious that in most domains of mathematics the exact nature of the mathematical objects studied is not essential. What is of more central concern are the patterns of thought applied by mathematicians in reaching new results, or by students in finding for themselves solutions of problems or proofs of known theorems.

As yet, theories of learning have little to offer in providing insight into how one learns to think mathematically. The nature of abstraction, or the processes of imagery and association that are surely essential to thinking in any domain of mathematics, have as yet scarcely been studied from a scientific standpoint.

Like mathematics itself, research in mathematics education will necessarily have both basic and applied components. Research that is concerned with particular pieces of curriculum and particular learning difficulties of students will continue to occupy a major portion of research efforts, but it is also to be hoped that the kind of problems I have just been mentioning, problems that represent fundamental puzzles about the nature of human thinking, will come to occupy a larger place in research about mathematics learning.