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Theoretical Note

The Definability of the Qualitative Independence of Events in Terms of Extended Indicator Functions

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The main purpose of this paper is to define the relation of probabilistic independence of events in terms of a qualitative ordering of extended indicator functions of events. © 1994 Academic Press, Inc

1. INTRODUCTION

An ordering $\succcurlyeq$ over events is often interpreted as a qualitative probabilistic ordering; that is, the intuitive sense of $A \succcurlyeq B$ is $P(A) \geqslant P(B)$, where P is a probability measure on the algebra of these events. There exists an extensive literature on examining the conditions that guarantee the existence of such a probabilistic representation for some set of ordered events (for a brief survey see, for example, Suppes and Zanotti, 1976). In this paper, we consider the qualitative relation of probabilistic independence $\perp$; the desired interpretation of $\perp$ is

$$A \perp B \qquad \text{iff} \quad P(A \cap B) = P(A)\, P(B). \tag{1}$$

The independence relation $\perp$ has been studied in the literature since at least Domotor (1969). Let Ω be the space of possible outcomes. Domotor gives a definition of $\perp$ in terms of a qualitative ordering on the Cartesian product of elements from $\mathscr{F}$, the algebra of subsets of Ω: $A \perp B$ iff $A \cap B \times \Omega \sim A \times B$ for all $A, B \in \mathscr{F}$, where $\sim$ is an equivalence relation induced by $\succcurlyeq$ and $A \times B \succcurlyeq C \times D$ iff

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$P(A)\,P(B) \geqslant P(C)\,P(D)$. He proved from this definition some basic properties of $\perp$. In Krantz, Luce, Suppes, and Tversky (1971) a four-place relation also was used to define independence. Another approach was taken by Fine (1973) who proposed a set of axioms for $\succcurlyeq$ and $\perp$, which are not sufficient to prove (1). For results on the independence of disjoint sets of random variables see Geiger, Paz, and Pearl (1991). Independence for such sets of random variables does not imply independence of individual events, and vice versa. For example, in the standard language of random variables one cannot express that $B \cap C$ is independent of A but $B \cap \neg C$ is not independent of A.

In this paper the relation of the probabilistic independence of events is defined in terms of the qualitative ordering of extended indicator functions of events. The restricted introduction of elementary random variables, i.e., indicator functions of events and their closure under addition, strengthens the concept of event just enough to define satisfactorily the qualitative independence of events. This is not surprising, for this same apparatus is the one used in Suppes and Zanotti (1976) to give simple necessary and sufficient conditions for numerical representations of a qualitative probabilistic ordering. The purely random-variable characterization in Geiger *et al.* (1991) and related literature serves another purpose, as does the substitution of an abstract linear space of random variables in place of the extended indicator functions, which do not form a linear space but only a weaker additive semigroup, introduced here.

2. Undefinability of $\perp$ in Terms of Qualitative Ordering over Events

Some natural elementary axioms on independence were given by Fine (1973):

I1. $A \perp B \Rightarrow B \perp A$;

I2. $A \perp \Omega$;

I3. $A \perp B \Rightarrow A \perp \neg B$;

I4. $B \cap C = \varnothing,\ A \perp B,\ A \perp C \Rightarrow A \perp (B \cup C)$;

I5. $A \perp B,\ C \perp D,\ B \succcurlyeq D \Rightarrow (A \succcurlyeq C \Rightarrow A \cap B \succcurlyeq C \cap D),\ (A \succ C,\ B \succ \varnothing \Rightarrow A \cap B \succ C \cap D)$.

He proved that these conditions guarantee the existence of a probability assignment P, agreeing with $\succcurlyeq$, and of an operation G such that

$$A \perp B \Rightarrow P(A \cap B) = G(P(A), P(B)).$$

In order to make G associative, Fine suggests the following axiom, where $\approx$ is the equivalence relation defined in terms of $\succcurlyeq$.

I6. $A \perp B \cap C,\quad B \perp C,\quad A' \cap B' \perp C',\quad A' \perp B',\quad A \approx A',\quad B \approx B',\quad C \approx C' \Rightarrow A \cap B \cap C \approx A' \cap B' \cap C'$.

Another property, not derivable from I1–I6, is

I7. $A \perp B,\ B \perp C,\ A \perp B \cap C \Rightarrow C \perp A \cap B.$

It is still an open question as to whether $\perp$ (interpreted as $A \perp B$ iff $P(A \cap B) = P(A) \cdot P(B)$) could be given a finite axiomatization. In Fine's system, the independence of events does not follow from $P(A \cap B) = G(P(A), P(B))$ (intuitively, from the product factorization of probability). Fine's axioms I1–I7 do not imply the distributive property of G over addition, i.e, they do not provide a numerical representation in which G stands for multiplication.

However, it is easy to show that the independence relation is undefinable in terms of events and a qualitative ordering over them, a result that is part of the folklore of the subject, but, not, as far as we know, written down anywhere.

THEOREM 1. *Let* $(\Omega, \succcurlyeq, \perp)$ *be such that*

1. Ω *is a finite nonempty set*;
2. $A \succcurlyeq B$ *or* $B \succcurlyeq A$;
3. $\succcurlyeq$ *is a relation on the subsets of* Ω *satisfying Scott's conditions*:

S1. $A \succcurlyeq \varnothing$;

S2. $\Omega \succ \varnothing$;

S3. *for all subsets* $A_0, ..., A_n, B_0, ..., B_n$ *of* Ω, *if* $A_j \succcurlyeq B_j$ *for* $0 \leqslant j < n$, *and for all* ω *in* Ω

$$A_0^{\iota}(\omega) + \cdots + A_n^{\iota}(\omega) = B_0^{\iota}(\omega) + \cdots + B_n^{\iota}(\omega),$$

then $A_n \preccurlyeq B_n$, *where* A^{ι} *is the indicator function of event* A, *i.e.*, $A^{\iota}(\omega) = 1$ *if* $\omega \in \Omega$ *and* $A^{\iota}(\omega) = 0$ *if* $\omega \notin \Omega$.

4. $\perp$ *is a relation satisfying Fine's axioms* I1–I7 *given above.*

Then the independence relation $\perp$ *is not definable in terms of the structure* $(\Omega, \succcurlyeq)$.

Proof. Consider the following structure: $\Omega = \{a, b, c, d\}$; $\succcurlyeq$ is defined as follows, where, by abuse of notation, a replaces $\{a\}$, $a+b$ replaces $\{a, b\}$, etc.; and $\prec$ is defined in the usual way in terms of $\succcurlyeq$:

$0 \prec a \prec b \prec c \prec a+b \prec a+c \prec d \prec b+c \prec a+d \prec a+b+c \prec b+d \prec c+d \prec a+b+d \prec a+c+d \prec b+c+d \prec a+b+c+d.$

Let event $A = \{a, b\}$, $B = \{b, d\}$. We show that this structure can be given two different numerical interpretations, both preserving the ordering, i.e., satisfying 1–3 above, but in one of them A and B are independent,

$$P(A \cap B) = P(A)\, P(B),$$

and in the second they are not; the notion of independence agrees with Fine's axioms.

$$P_1: a=10/80, b=15/80, c=22/80, d=33/80; a+b+c+d=1.$$

$$P_2: a=10/80, b=16/80, c=21/80, d=33/80.$$

In the first case $15/80 = 25/80 \cdot 48/80$, in the second $16/80 \neq 26/80 \cdot 49/80$. Whence by Padoa's principle $\perp$ is not definable in terms of $(\Omega, \succcurlyeq)$. (Padoa's principle is this: to prove that a concept is independent of, i.e., not definable in terms of, a set of given concepts, it is sufficient to find two models that are the same for the given concepts but different for the concept in question.) Q.E.D.

3. Definition of Independence in Terms of Extended Indicator Functions

The notion of extended indicator function was used in Suppes and Zanotti (1976) to give a simple set of conditions for the existence of a probability measure agreeing with the qualitative ordering.

Let Ω be the set of possible outcomes, $\mathscr{F}$ an algebra of events on Ω. Let A' be the indicator function (or characteristic function) of event A. The algebra $\mathscr{F}^*$ of extended indicator functions relative to $\mathscr{F}$ is the smallest semigroup (under functional addition) containing the indicator functions of all events in $\mathscr{F}$. In other words, $\mathscr{F}^*$ is the smallest set of functions such that (i) if $A \in \mathscr{F}$ then $A' \in \mathscr{F}^*$, and (ii) if $A^*, B^* \in \mathscr{F}^*$ then $A^* + B^* \in \mathscr{F}^*$.

It is easy to see that extended indicator functions are elementary random variables of a restricted sort (taking nonnegative integer values). The intuitive sense of qualitative ordering is that $A^* \succcurlyeq B^*$ ($A^*, B^* \in \mathscr{F}^*$) iff the expected value of A^* is greater or equal to the expected value of B^*. Certainly, for $A, B \in \mathscr{F}$, $A' \succcurlyeq B'$ iff $A \succcurlyeq B$. (Note that here and later we use the same symbols for the ordering relation for events and for extended indicator functions. A similar remark applies to $\succ$, $\preccurlyeq$, $\prec$, $\approx$, all definable in terms of $\succcurlyeq$.)

Definition 1 (Suppes & Zanotti, 1976). Let Ω be a nonempty set, let $\mathscr{F}$ be an algebra of sets on Ω, and let $\succcurlyeq$ be a binary relation on $\mathscr{F}^*$, the algebra of extended indicator functions relative to $\mathscr{F}$. Then the qualitative algebra $(\Omega, \mathscr{F}^*, \succcurlyeq)$ is qualitatively satisfactory if and only if the following axioms are satisfied for every A^*, B^*, and C^* in $\mathscr{F}^*$.

Axiom 1. The relation $\succcurlyeq$ is a weak ordering of $\mathscr{F}^*$.

Axiom 2. $\Omega' \succ \varnothing'$.

Axiom 3. $A^* \succcurlyeq \varnothing'$.

Axiom 4. $A^* \succcurlyeq B^*$ iff $A^* + C^* \succcurlyeq B^* + C^*$.

Axiom 5. If $A^* \succ B^*$ then for every C^* and D^* in $\mathscr{F}^*$ there is a positive integer n such that $nA^* + C^* \succcurlyeq nB^* + D^*$.

(Note that nA^* stands for $A^* + \cdots + A^*$ n times).

It was proved in Suppes and Zanotti (1976) that a necessary and sufficient condition for existence of a strictly agreeing probability measure on $\mathscr{F}$ is that there is an extension of $\succcurlyeq$ from $\mathscr{F}$ to $\mathscr{F}^*$ such that $(\Omega, \mathscr{F}^*, \succcurlyeq)$ is qualitatively satisfactory and this measure is unique when extended to $\mathscr{F}^*$. However, in their representation theorem and others of a like nature the requirement that the measure P of probability norm to 1 has a arbitrary character; i.e., the form of the representation permits multiplication by an arbitrary positive constant. In the case of Suppes and Zanotti (1976), the representation theorem is formulated in terms of an expectation function E for extended indicator functions satisfying the following obvious properties for any A^* and B^* in $\mathscr{F}^*$:

E1. $E(A^*) \geqslant E(B^*)$ iff $A^* \succcurlyeq B^*$

E2. $E(A^* + B^*) = E(A^*) + E(B^*)$.

It is clear that if a function E defined on $\mathscr{F}^*$ satisfies E1 and E2, so will αE, where α is a positive real number. On the other hand, the one "universal" positive probability case of independence, namely, the independence of Ω with respect to Ω, can be formulated as an additional requirement on E, and this additional requirement fixes E uniquely:

E3. $E((\Omega \cap \Omega)^\iota) = E(\Omega^\iota) \cdot E(\Omega^\iota)$.

From the axioms of Definition 1, it easily follows that $E(\varnothing^\iota) = 0$, $E(\Omega^\iota) > 0$, and thus from E3 that $E(\Omega^\iota) = 1$. We then also have a nonarbitrary norming of the probability measure: for A in $\mathscr{F}$,

E4. $P(A) = E(A^\iota)$.

The preceding remarks are necessary to avoid any air of paradox in our definition of independence of events within the framework of extended indicator functions, because a satisfactory definition of independence for which one can prove

$$A \perp B \quad \text{iff} \quad P(A \cap B) = P(A)\,P(B)$$

clearly requires that the probability measure be normed to 1. It would be paradoxical if this could be done only after the definition of $\perp$. As the above considerations show, this is not the case.

These remarks establish the following representation theorem, which is stronger than the one given in Suppes and Zanotti (1976).

THEOREM 2. *If the structure* $(\Omega, \mathscr{F}, \succcurlyeq)$ *satisfies Definition* 1, *then there is a unique expectation function E on $\mathscr{F}^*$ and a unique probability measure P which together satisfy* E1–E4.

Within the framework of Definition 1, we now define independence as follows.

DEFINITION 2. $A \perp B$ iff

a. $\forall m \forall m' \forall n \forall n' (m\Omega' \preccurlyeq nA' \;\&\; m'\Omega' \preccurlyeq n'B' \Rightarrow mm'\Omega' \preccurlyeq nn'(A \cap B)')$

b. $\forall m \forall m' \forall n \forall n' (m\Omega' \succcurlyeq nA' \;\&\; m'\Omega' \succcurlyeq n'B' \Rightarrow mm'\Omega' \succcurlyeq nn'(A \cap B)')$.

It was proved in Suppes *et al.* (1989) (based on Suppes & Zanotti, 1991) (for a bit different setup) that the conditions of the left side of the definition guarantee that this property holds. We follow this proof here.

THEOREM 3. *Given that* $(\Omega, \mathscr{F}, \succcurlyeq)$ *satisfies Definition* 1 *with a unique expectation function* E *and that measure* P *satisfies* E1–E4, *then for* A, B *in* $\mathscr{F}$

$$P(A \cap B) = P(A)\,P(B) \qquad \textit{iff} \quad A \perp B.$$

Proof. To prove the theorem, we define, for every A' in $\mathscr{F}^*$, the set S_A of numbers,

$$S_A = \left\{\frac{m}{n} : m\Omega' \succcurlyeq nA'\right\}.$$

It is easy to show that S_A is nonempty and has a greatest lower bound, which is in fact just the unique expectation function E:

$$E(A') = \text{glb}\, S_A.$$

For independent A and B we have (see Definition 2a) $m/n \in S_A$, $m'/n' \in S_B \Rightarrow mm'/nn' \in S_{A \cap B}$, and thus

$$E((A \cap B)') \leqslant E(A')\,E(B').$$

Then we define $T_A = \{m/n : m\Omega' \preccurlyeq nA'\}$. Each set T_A is obviously nonempty and has a least upper bound. We need to show that

$$\text{lub}\, T_A = \text{glb}\, S_A = E(A').$$

Suppose, by way of contradiction, that

$$\text{lub}\, T_A < \text{glb}\, S_A$$

(it is evident that lub $T_A \leqslant$ glb S_A holds). Then there must exist integers m and n such that

$$\text{lub}\, T_A < \frac{m}{n} < \text{glb}\, S_A.$$

Thus, we have from the left-hand inequality

$$m\Omega' \succ nA',$$

and from the right-hand one

$$m\Omega^{\iota} \prec nA^{\iota},$$

which together contradict the order properties of $\succcurlyeq$.

Analogously with the case of S_A, the definition of T_A implies

$$E(A^{\iota})\, E(B^{\iota}) \leqslant E((A \cap B)^{\iota}),$$

and together they give

$$E(A^{\iota})\, E(B^{\iota}) = E((A \cap B)^{\iota}). \qquad \text{Q.E.D.}$$

4. Some Properties of $\perp$

Now we show that Fine's axioms I1–I7 are derivable in an elementary way.

THEOREM 4. *Fine's axioms on independence* I1–I7 *are derivable from Axioms* 1–5 *of Definition* 1 *and the definition of independence.*

Proof. The proof of I1–I3 is trivial.

I4. If $B \cap C = \varnothing$, then $B^{\iota} + C^{\iota} = (B \cup C)^{\iota}$ and

$$(A \cap B)^{\iota} + (A \cap C)^{\iota} = ((A \cap B) \cup (A \cap C))^{\iota}.$$

We should prove that $\forall m\, \forall m^*\, \forall n\, \forall n^* (m\Omega^{\iota} \succcurlyeq nA^{\iota}\ \&\ m^*\Omega^{\iota} \succcurlyeq n^*(B \cup C)^{\iota} \Rightarrow mm^*\Omega^{\iota} \succcurlyeq nn^*(A \cap (B \cup C))^{\iota})$. (And do the same for $\preccurlyeq$.)

Let $m\Omega^{\iota} \succcurlyeq nA^{\iota}$ and $m^*\Omega^{\iota} \succcurlyeq n^*(B \cup C)^{\iota}$. Since B and C are disjoint,

$$m^*\Omega^{\iota} \succcurlyeq n^*(B^{\iota} + C^{\iota})$$

and

$$m^*\Omega^{\iota} \succcurlyeq n^*B^{\iota} + n^*C^{\iota}.$$

There exist m', m'' such that $m' + m'' = m^*$ and

$$m'\Omega \succcurlyeq n^*B^{\iota}, \qquad m''\Omega \succcurlyeq n^*C^{\iota}$$

(from the properties of $\succcurlyeq$ and $+$). Since $A \perp B$ and $A \perp C$,

$$mm'\Omega \succcurlyeq nn^*(A \cap B)^{\iota}$$

and

$$mm''\Omega \succcurlyeq nn^*(A \cap C)^{\iota},$$

thus

$$m(m' + m'')\Omega \succcurlyeq nn^*((A \cap B)^{\iota} + (A \cap C)^{\iota}).$$

This gives us

$$mm^*\Omega \succcurlyeq nn^*((A \cap B) \cup (A \cap C))'.$$

From Boolean algebra we obtain

$$mm^*\Omega \succcurlyeq nn^*(A \cap (B \cup C))'.$$

The proof for $\preccurlyeq$ is analogous.

Before proving I5 we need to prove the following.

LEMMA. *If $A \perp B$, then*

I5′. $A \perp B \Rightarrow \forall m\, \forall m'\, \forall n\, \forall n'(m\Omega' \succcurlyeq nA'\ \&\ m'\Omega' \succ n'B' \Rightarrow mm'\Omega' \succ nn'(A \cap B)')$;

I5″. $A \perp B \Rightarrow \forall m\, \forall m'\, \forall n\, \forall n'(m\Omega' \preccurlyeq nA'\ \&\ m'\Omega' \prec n'B' \Rightarrow mm'\Omega' \prec nn'(A \cap B)')$.

Proof. I5′. Let $A \perp B$. Then

1. $\forall m\, \forall m'\, \forall n\, \forall n'(m\Omega' \succcurlyeq nA'\ \&\ m'\Omega' \succcurlyeq n'B' \Rightarrow mm'\Omega' \succcurlyeq nn'A \cap B')$;
2. let $m\Omega' \succcurlyeq nA'$;
3. let $m'\Omega' \succ n'B'$;
4. $\exists n^*(n^*m'\Omega' \succcurlyeq n^*n'B' + B')$ from 3 and Axiom 5;
5. $n^*m'\Omega' \succcurlyeq (n^*n' + 1)\, B'$ from 4;
6. $mn^*m'\Omega' \succcurlyeq n(n^*n' + 1)(A \cap B)'$ from 1, 2, and 5;
7. $mn^*m'\Omega' \succcurlyeq nn^*n'(A \cap B)' + n(A \cap B)'$ from 6;
8. $mn^*m'\Omega' \succ nn^*n'(A \cap B)'$ from 7;
9. $mm'\Omega' \succ nn'(A \cap B)'$ from 8.

I5″. The proof is analogous.

Now we continue the proof of Theorem 4.

I5. We consider two cases:

1. $A' \succ C'$, $B' \succ D'$: it is evident that $\exists m\, \exists m'\, \exists n\, \exists n'(m\Omega' \prec nA', m\Omega' \succcurlyeq nC', m'\Omega' \prec n'B', m'\Omega' \succcurlyeq n'D')$; thus, for some mm' and nn', $nn'(A \cap B)' \succ mm'\Omega' \succcurlyeq nn'(C \cap D)'$, and this means that $(A \cap B)' \succ (C \cap D)'$.

2. $A' \approx C'$, $C' \succ \varnothing'$, $B' \succ D'$: there are m, n such that

$$nA' \prec m\Omega',\ (n+1)\, A' \succcurlyeq m\Omega',$$

and the same should hold for C'. Since $B' \succ D'$, there exist such n', m' that $n'B' \succ m'\Omega' \succcurlyeq (n' + n')\, D'$. Now we have

$$n'(n+1)(A \cap B)' \succ mm'\Omega' \succ n(n' + n')(C \cap D)',$$

or

$$n'n(A \cap B)^{\iota} + n'(A \cap B)^{\iota} \succ n'n(C \cap D)^{\iota} + nn'(C \cap D)^{\iota}.$$

Since $n' \preccurlyeq nn'$, $(A \cap B)^{\iota} \succ (C \cap D)^{\iota}$.

I6. Let $m\Omega^{\iota} \succcurlyeq nA^{\iota}$, $m'\Omega^{\iota} \succcurlyeq n'B^{\iota}$, $m''\Omega^{\iota} \succcurlyeq n''C^{\iota}$. Since $A \approx A'$, $B \approx B'$, $C \approx C'$, the same holds for A'^{ι}, B'^{ι}, and C'^{ι}. Now, $m'm''\Omega^{\iota} \succcurlyeq n'n''(B \cap C)^{\iota}$ (because $B \perp C$) and $mm'm''\Omega^{\iota} \succcurlyeq nn'n''(A \cap B \cap C)^{\iota}$ (because $A \perp B \cap C$). Since $A' \perp B'$, $mm'\Omega^{\iota} \succcurlyeq nn'(A' \cap B')^{\iota}$, and since $A' \cap B' \perp C'$, $mm'm''\Omega^{\iota} \succcurlyeq nn'n''(A' \cap B' \cap C')^{\iota}$. We obtain

$$\forall m\, \forall m'\, \forall m''\, \forall n\, \forall n'\, \forall n''(m\Omega^{\iota} \succcurlyeq nA^{\iota}\ \&\ m'\Omega^{\iota} \succcurlyeq n'B^{\iota}\ \&\ m''\Omega^{\iota} \succcurlyeq n''C^{\iota}$$

$$\Rightarrow (mm'm''\Omega^{\iota} \succcurlyeq nn'n''(A \cap B \cap C)^{\iota}\ \&\ mm'm''\Omega^{\iota} \succcurlyeq nn'n''(A' \cap B' \cap C')^{\iota})).$$

It is evident that if $A \cap B \cap C \prec A' \cap B' \cap C'$, there exist m^*, n^* such that $n^*(A \cap B \cap C)^{\iota} \preccurlyeq m^*\Omega^{\iota} \prec n^*(A' \cap B' \cap C')^*$. We can always find some k such that km^* and kn^* could be represented as multiplications of m, m', m'' and n, n', n'', respectively, for which hold $m\Omega^{\iota} \succcurlyeq nA^{\iota}$, $m'\Omega^{\iota} \succcurlyeq n'B^{\iota}$, $m''\Omega^{\iota} \succcurlyeq n''C^{\iota}$. But, as we have proved, this means that $mm'm''\Omega^{\iota} \succcurlyeq nn'n''(A' \cap B' \cap C')^{\iota}$, which contradicts the assumption. The same argument works for $A \cap B \cap C \succ A' \cap B' \cap C'$. Thus, $(A \cap B \cap C)^{\iota} \approx (A' \cap B' \cap C')^{\iota}$.

I7. We need to prove that

$$\forall m''\, \forall m^*\, \forall n''\, \forall n^*(m''\Omega^{\iota} \succcurlyeq n''C^{\iota}\ \&\ m^*\Omega^{\iota} \succcurlyeq n^*(A \cap B)^{\iota}$$

$$\Rightarrow m''m^*\Omega^{\iota} \succcurlyeq n''n^*(A \cap B \cap C)^{\iota}),$$

and the same for $\preccurlyeq$.

Assume that for some m'', n'', m^*, n^* this does not hold. This means that $m''\Omega^{\iota} \succcurlyeq n''C^{\iota}\ \&\ m^*\Omega^{\iota} \succcurlyeq n^*(A \cap B)^{\iota}\ \&\ m''m^*\Omega^{\iota} \prec n''n^*(A \cap B \cap C)^{\iota}$.

As in the previous proof, we state first that if $m^*\Omega^{\iota} \succcurlyeq n^*(A \cap B)^{\iota}$, then there is some k such that $km^* = mm'$, $kn^* = nn'$, and $m\Omega^{\iota} \succcurlyeq nA^{\iota}$, $m'\Omega^{\iota} \succcurlyeq n'B^{\iota}$. From this we get

$$mm'm''\Omega^{\iota} \succcurlyeq nn'n''(A \cap B \cap C)^{\iota}$$

$$km^*m''\Omega^{\iota} \succcurlyeq kn^*n''(A \cap B \cap C)^{\iota}$$

$$m^*m''\Omega^{\iota} \succcurlyeq n^*n''(A \cap B \cap C)^{\iota},$$

which contradicts our assumption.

After we repeat the same argument for $\preccurlyeq$, we get both conditions for $C \perp A \cap B$.

Q.E.D.

5. Further Developments

We mention some open problems and further possible developments.

Finite axiomatizability of $\perp$. We have in mind, of course, finite axiomatizability in first-order logic. It seems likely the result will be negative, and even more so for axiomatizable by a universal sentence, if we want necessary and sufficient conditions for any finite Boolean algebra. However, some interesting sufficient conditions might be given for significant classes of qualitative probability relations. For example, in the special but significant finite case of all atoms being equiprobable, a non-first-order definition of independence can be easily given. Let $|A|$ be the cardinality of set A. Then

$$A \perp B \quad \text{iff} \quad |\Omega| \cdot |A \cap B| = |A| \cdot |B|.$$

Positive relevance. It is obvious that defining probabilistic independence is closely connected with defining the notion of positive relevance ($P(A \cap B) > P(A)\,P(B)$). Theorem 1 establishes that the notion of positive relevance is undefinable in terms of $(\Omega, \succcurlyeq)$, since independence itself is definable via positive relevance.

Let $\top$ stand for relevance, $\heartsuit$ for positive relevance, and $\spadesuit$ for negative relevance. Then

$$A \perp B \quad \text{iff } \textit{not} \quad (A \top B)$$

$$A \top B \quad \text{iff} \quad A \heartsuit B \quad \textit{or} \quad A \spadesuit B$$

$$A \spadesuit B \quad \text{iff} \quad A \heartsuit \neg B$$

$$A \perp B \quad \text{iff } \textit{not} \quad (A \heartsuit B \textit{ or } A \heartsuit \neg B).$$

So, if independence is not definable neither is positive relevance. The reverse does not hold: positive relevance is not definable via independence (although relevance is). We can give positive relevance the following definition in terms of extended indicator functions.

$A \heartsuit B$ iff

a. $\exists m\, \exists m'\, \exists n\, \exists n'(m\Omega' \succcurlyeq nA' \;\&\; m'\Omega' \succcurlyeq n'B' \;\&\; mm'\Omega' \prec nn'(A \cap B)')$

b. $\forall m\, \forall m'\, \forall n\, \forall n'(m\Omega' \preccurlyeq nA' \;\&\; m'\Omega' \preccurlyeq n'B' \Rightarrow mm'\Omega' \prec nn'(A \cap B)')$.

Unlike independence, it requires existential quantifiers. Questions about axiomatizability of positive relevance, similar to those for independence, are apparantly also as yet unanswered.

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