The Principle of Invariance with Special Reference to Perception

*Patrick Suppes**
Stanford University

The principle of invariance is now a familiar one in psychology, especially because of its prominent role in the theory of measurement. The relation of invariance to the meaningfulness of various statistics for various scales of measurement is a particularly salient example that has received much discussion in the literature for over thirty years. Perhaps the most notorious case is that of whether standard intelligence tests have more than ordinal properties.

The concept of invariance has an older history in geometry. It came to prominence in the nineteenth century with the work of Felix Klein and his view that the transformations themselves could be taken as primitive geometrical concepts. For example, Euclidean geometrical properties are characterized by their invariance under the group of Euclidean motions. There is, however, a lot more to geometry than Kleinian-type transformations, but the principle of invariance remains an important one.

The principle has also achieved importance in physics, and in many ways has had a more prominent role in the twentieth century in physics than in geometry. Both invariance under the Galilean transformations of classical physics, the Lorentz transformations of special relativity, and the transformations of general relativity have been central ideas in modern physics. But the central role of invariance does not stop there. Of almost equal importance is the relation between a given invariance or symmetry property of a physical system and a corresponding conservation law. Such relations were first studied by Emmy Noether, and now many Noetherian theorems play an important role in physical theories, for example, theorems on the conservation of energy, momentum and angular momentum in both classical and quantum mechanics.

My purpose in this lecture is to examine the role of the principle of invariance in perception, which moves away from the concerns of invariance in measurement, as do the Noetherian theorems in physics. On the other hand, I shall not attempt anything like a systematic survey, for the ways in which concepts of invariance enter into perception are many and varied. Rather, I shall focus on two main areas, both concerned primarily with visual perception. The first is the kind of invariance related to ordinary experience and the use of ordinary language. Here I shall be especially concerned with the relation between perception and geometry, less so with the ways

*I am indebted to Michael Böttner for a number of helpful comments. Correspondence should be addressed to Patrick Suppes, Ventura Hall, Stanford University, Stanford, CA 94305-4115.

in which physical concepts also influence, or are part of the meaning of, ordinary perceptual descriptions. In the second part I examine in some detail whether anything like the invariance characteristic of the physical concepts of space or space and time, as reflected in Galilean or Lorentz invariance, can be expected to hold for the perception of visual space. And if not, are there other, perhaps more limited, principles of symmetry that are salient.

A preliminary remark is needed about the way the principle of invariance enters the analyses I am concerned with. In most of the literature in psychology or physics on invariance, a given theory is held fixed and unchanged, and the invariance of some relation, function or proposition with respect to the theory is examined. For example, in a purely ordinal theory of measurement the mean of a set of numerical data is not invariant with respect to the theory, for arbitrary monotone transformations of the numerical data are permitted by a purely ordinal theory, and the mean is obviously not invariant under an arbitrary monotone transformation. Here I pursue a different path. In Part I various ordinary spatial expressions are introduced with their ordinary meaning, and the question asked is this. What is the natural geometrical theory with respect to which each expression is invariant? In Part II, similar questions are asked about the results of various experiments on visual space which challenge the thesis that visual space is Euclidean. As already hinted at, I also shall follow the practice in geometry and physics of relating invariance under a given group of transformations to a principle of symmetry. Invariance implies symmetry and symmetry implies invariance.

1 Geometry of Qualitative Visual Perceptions as Expressed in Ordinary Language

The concepts, results and problems that fall under the general heading of this part of the lecture have not been studied very intensely in the past but have received more attention in the last ten years or so. I mention in this connection especially Bowerman (1989), Crangle and Suppes (1989), and Levelt (1982, 1984). The references in these publications provide good leads back into the older literature. A typical problem that I have in mind here is the sort of thing that Levelt has investigated thoroughly. For example, the way language is used to give spatial directions, and the limitations of our ability to give such directions or describe visual scenes with accuracy. In the case of my own earlier work with Crangle, we were concerned especially to analyze the geometry underlying the use of various common prepositions in English.

The results of that analysis can be easily summarized in Table 1 which is reproduced with some modifications from Crangle and Suppes (1989). The kinds of geometry referred to are standard, with the exception perhaps of the geometry of oriented physical space. For example, in the case of the sentence *The pencil is in the box* (where it is assumed the box is closed), it is clear that only a purely topological notion of invariance is needed. On the other hand, in the sentence *Mary is sitting between Jose and Maria*, it is easy to see that the geometry needs to be affine. And once the idea of a metric is introduced, as in the sentence *The pencil is near the box*, we must go from affine geometry to some underlying notion of congruence as reflected in

Euclidean geometry. Although we may be more refined in the analysis, note that in this sentence it is quite satisfactory to use absolute geometry which is Euclidean geometry minus the standard Euclidean axiom. This axiom asserts that given a point a and a line L on which the point does not lie, then there exists at most one line through a in the plane formed by aL which does not meet the line. We get hyperbolic rather than Euclidean geometry by adding the negation of this axiom to absolute geometry. It seems to me that the notion of nearness used in ordinary talk is satisfied well enough by the congruence relation of absolute geometry—a still weaker geometry of rigid bodies is considered later. In fact, relatively technical geometrical results are required to move us from absolute to Euclidean geometry, the sort of technical facts required, for example, in architecture and construction. Another way of noting the adequacy of absolute geometry to express many of the elementary results of Euclidean geometry is that the first 26 propositions of Book I of Euclid's *Elements* are provable in absolute geometry. On the other hand, in the case of the preposition *on*, it is clear that a notion of vertical orientation is required, a notion completely absent from Euclidean geometry, and in fact not definable within Euclidean geometry. A different kind of orientation is required in the case of objects that have a natural intrinsic orientation. Consider for instance, the sentence given in Table 1, *The dog is in front of the house.* Finally, in the case of many processes it is not sufficient to talk about static spatial geometry but for a full discussion one needs the assumption of space-time. An example is given at the end of Table 1.

Table 1. *Kinds of Geometry and Examples of Prepositional Use.*

Topology	*The pencil is in the box. (box closed)* *One piece of rope goes over and under the other.*
Affine geometry	*The pencil is in the box. (box open)* *Mary is sitting between Jose and Maria.*
Absolute geometry	*The pencil is near the box*
The geometry of oriented physical space	*The book is on the table.* *Adjust the lamp over the table.*
Projective geometry	*The post office is over the hill.* *The cup is to the left of the plate.*
Geometries that include figures and shapes with orienting axes	*The dog is in front of the house.* *The pencil is behind the chair.*
Geometry of classical space-time	*She peeled apples in the kitchen.*

What I now want to look at are the kinds of axioms needed to deal with the cases of geometry that are not standard. It would not be appropriate simply to repeat standard axioms for topological, projective, affine, absolute, and Euclidean geometry. A rather thorough recent presentation of these geometries is to be found in Suppes, et. al., (1989, Ch. 13). What I shall do is make reference to the primitive notions on which these various axioms are based as given in *Foundations of Measurement, Volume II.*

Oriented Physical Space. Undoubtedly, the aspect of absolute or Euclidean geometry which most obviously does not satisfy ordinary talk about spatial relations is that there is no concept of vertical orientation. Moreover, on the basis of well-known results of Tarski concerning the fact that no nontrivial binary relations can be defined in Euclidean geometry, the concept is not even definable in Euclidean geometry. For definiteness I have in mind as the primitive concepts of Euclidean geometry the affine ternary relation of betweenness for points on a line and the concept of congruence for line segments. In many ways I should mention however, it is more convenient to use the notion of parallelism and perpendicularity, and in any case I shall assume these latter two notions are defined. There are many different ways, of course, of axiomatizing as an extension of Euclidean geometry the concept of verticality. One simple approach is to add as a primitive the set $\mathcal{V}$ of vertical lines, and then to add axioms of the following sort to three-dimensional Euclidean geometry. *Given any point a there is a vertical line through a. If K is a vertical line and L is parallel to K, then L is a vertical line.*

There are however, difficulties with this approach of two different kinds. The first and conceptually the most fundamental is that our natural notion of oriented physical space as we move around in our ordinary environment is that the orientation of the space, both vertically and horizontally, is fixed uniquely. We do not have arbitrary directional transformations of verticality nor of horizontal orientation. We naturally have a notion of north, east, south, and west, with corresponding degrees. Secondly, and closely related to this, very early in the discussion of the nature of physical space, it was recognized that we have difficulties with treating the surface of the earth as a plane with horizontal lines being in this plane and vertical lines being perpendicular to them. The natural geometry of oriented physical space in terms of ordinary experience — a point to be expanded upon in a moment —, is in terms of spherical geometry, with the center of the earth being an important concept, as it was for Aristotle and Ptolemy in ancient times, who in many ways most clearly expressed ideas about the nature of physical space.

Aristotle directly uses perceptual evidence as part of his argument for the conclusion that the earth is spherical *On The Heavens* Book II, Ch. 14, 297b): if the earth were not spherical, eclipses of the moon would not exhibit the shapes they do, and observation of the stars would not show the variation they do as we move to the north or south.

Ptolemy's argument in the *Almagest* is even better and because it will not be familiar to many readers, I quote in full, for the arguments are perceptual throughout. This passage occurs in Book I, Section 4 of the *Almagest*, written more than four hundred years later than Aristotle's work.

That the earth, too, taken as a whole, is sensibly spherical can best be grasped from the following considerations. We can see, again, that the sun, moon and other stars do not rise and set simultaneously for everyone on earth, but do so earlier for those more towards the east, later for those towards the west. For we find that the phenomena at eclipses, especially lunar eclipses, which take place at the same time [for all observers], are nevertheless not recorded as occurring at the same hour (that is at an equal distance from noon) by all observers. Rather, the hour recorded by the more easterly observers is always later than that recorded by the more westerly. We find that the differences in the hour are proportional to the distances between the places [of observation]. Hence one can reasonably conclude that the earth's surface is spherical, because its evenly curving surface (for so it is when considered as a whole) cuts off [the heavenly bodies] for each set of observers in turn in a regular fashion.

If the earth's shape were any other, this would not happen, as one can see from the following arguments. If it were concave, the stars would be seen rising first by those more towards the west; if it were a plane, they would rise and set simultaneously for everyone on earth; if it were triangular or square or any other polygonal shape, by a similar argument, they would rise and set simultaneously for all those living on the same plane surface. Yet it is apparent that nothing like this takes place. Nor could it be cylindrical, with the curved surface in the east-west direction, and the flat sides towards the poles of the universe, which some might suppose more plausible. This is clear from the following: for those living on the curved surface none of the stars would be ever-visible, but either all stars would rise and set for all observers, or the same stars, for an equal [celestial] distance from each of the poles, would always be invisible for all observers. In fact, the further we travel toward the north, the more of the southern stars disappear and the more of the northern stars appear. Hence it is clear that here too the curvature of the earth cuts off [the heavenly bodies] in a regular fashion in a north-south direction, and proves the sphericity [of the earth] in all directions.

There is the further consideration that if we sail towards mountains or elevated places from and to any direction whatever, they are observed to increase gradually in size as if rising up from the sea itself in which they had previously been submerged: this is due to the curvature of the surface of the water. (pp. 40-41)

Aristotle's concept of natural motion, which means that heavy bodies fall toward the center of the earth, is well argued for again by Ptolemy in Section 7 of Book I.

...the direction and path of the motion (I mean the proper, [natural] motion) of all bodies possessing weight is always and everywhere at right angles to the rigid plane drawn tangent to the point of impact. It is clear from this fact that, if [these falling objects] were not arrested by the surface of the earth, they would certainly reach the

> center of the earth itself, since the straight line to the center is also always at right angles to the plane tangent to the sphere at the point of intersection [of that radius] and the tangent. (pp. 43-44)

What is important to notice about this argument and similar but less clear arguments of Aristotle is that the notion of vertical or up is along radii extended beyond the surface of the earth, and not in terms of lines perpendicular to one given horizontal plane. Thus the proper perceptual notion of verticality is in terms of a line segment that passes through the center of the earth. The notion of horizontal is that of a plane perpendicular to a vertical line at the surface of the earth. What is also important here is that the strongest argument for this viewpoint is the perceptual evidence of the nature of natural falling motion of heavy bodies.

Ptolemy also uses observations of the motion of the stars and the planets to fix the direction of east and west, and also the poles of north and south. We use in ordinary experience such arguments as the rising and setting of the sun to fix the direction of east and west, and in a similar vein we fix the north and south poles.

Looked at from the standpoint of the symmetries or invariance of the standard group of Euclidean motions, these perceptual arguments about physical space — including of course perceptual arguments about gravity —, reduce the group of Euclidean motions to the trivial automorphism of identity. This means that from a global standpoint the concept of invariance is not of any real significance in considering the perceptual aspects of oriented physical space. On the other hand the strong sense of the concept of global that is being used here must be emphasized. From the standpoint of the way the term *global* is used in general, there remain many symmetries of a less sweeping sort that are important and that are continually used in perceptual or physical analysis of actual space. Indeed this is a very intuitive outcome from the standpoint of ordinary perception. Combining both our visual sense of space and our sense of space arising from gravitational effects, it is wholly unnatural to think of anything like the group of Euclidean motions being the fundamental group for physical space in the sense of direct experience or perception. It is in fact a remarkable abstraction of the Greeks that orientation was not made a part of the original axioms of Euclid.

On the other hand, the notion of invariance in perception arises continually in a less global fashion in considering the symmetry of perceived figures or perceived phenomena arising from not just visual but also auditory or haptic data. A thorough development of symmetry in regular figures, especially two-dimensional ones, is given in Toth (1964). This is a subject with a rich history in both mathematics and arts, but I turn aside from it here to consider from a similar but different viewpoint the concept of invariance for what is expressed by a spatial preposition.

The classification of geometries for such prepositions given in Table 1 may be regarded as an *external* view from the standpoint of space as a whole. Perceptually, however, this is not the way we deal with the matter. In ordinary experience, we begin with the framework of a fixed oriented physical space as described above. Within this framework we can develop an *internal* view of spatial relations as expressed in ordinary language.

I first examine the invariance of the preposition *in*. In this and subsequent analysis the relation expressed by *in* shall be restricted to pairs of rigid bodies. Even though it is ultimately important to characterize invariance for other situations, e.g., the sugar being in the water, a good sense of the fundamental approach to invariance being advocated can be conveyed with consideration only of familiar situations with rigid bodies.

Let a and b be rigid bodies with a in b. As in Table 1 there are two cases of body b to consider. The first is when b has a closed hollow interior and a is in this interior. For simplicity of notation and analysis, I introduce the restrictive assumption that a is a spherical ball, so that the orientation of a can be ignored. Define $I(a,b)$ = the set of points in the closed hollow interior of b that the center of a can occupy. Let $\Phi(a,b)$ be the set of all transformations of $I(a,b)$ onto itself that represent possible changes in the position of a relative to b. For example, if the position of the center of a is on one side of the hollow interior of b it could be transformed, i.e., moved, to the other side of the interior without affecting the truth of the assertion that a is in b.

The set $\Phi(a,b)$ is the symmetry group (under the operation of function composition) of the position of a inside b, but because the hollow interior of b can be quite irregular in shape $\Phi(a,b)$ may not have standard properties of Euclidean isometries. It does have the expected invariance properties, namely, if $\varphi \in \Phi(a,b)$, then a with center c at point p is in b if and only if a with center c at point $\varphi(p)$ is in b.

Body b is itself subject to rigid motion in the fixed physical space. If b is closed, rotations around a horizontal axis are permitted, but if b is open, possible rigid motions of b must be restricted to those that preserve vertical orientation. And, of course, when b is subject to a rigid motion ψ, body a must be subject to the same transformation ψ, in order to preserve the invariance of the relation of being inside. Obviously, a may also be transformed by $\varphi \in \Phi(a,b)$, so that its full transformation could be the composition $\varphi \circ \psi$ with invariance of *in* preserved.

I turn now to a more extended analysis of the preposition *near*. Unfortunately, the only part of geometry that seems able to deal directly with natural objects taken as primitives is topology. Threads, knots, mazes and holes, for example, can be analyzed from a topological standpoint with considerable thoroughness, but as illustrated in Table 1, topology will not suffice even for the external analysis of many of the most common spatial prepositions such as *near*. As the sample sentences in Table 1 make plain, what at first seems to be most needed is the development of geometry based on the primitive concept of an approximately rigid body, but, as we shall see, difficulties lie in wait for this approach as well. A very preliminary and far from adequate analysis is given in Suppes (1972). It is easy to add to this earlier analysis a qualitative relation of distance between rigid bodies. The relation $ab \succeq cd$ is interpreted as the (qualitative) distance between bodies a and b being at least as great as the distance between bodies c and d. This distance is more specifically interpreted as being the minimum distance between two bodies, not the distance between their geometrical centers, which would be an alternative.

It is also feasible to introduce the affine relation of betweenness, $B(a, b, c)$. Here our intuitive qualitative judgment of body b being between bodies a and c is that something like the affine-point relation of betweenness holds for the centers of the bodies. Of course, the technical notion of the geometrical or gravitational center is not formally introduced. It is rather that we do seem to have an intuitive notion of the center of a body. Finally, the notion of one body being *near* another is relativized to context by introducing a pair of bodies a_0 and b_0 that serve as a standard in a given context. To say that two automobiles are near each other is in metric terms very different from saying that two books are near each other on the table. It is obvious that the standard for nearness changes from one context to another. Introducing the pair a_0 and b_0 is a formal device for dealing with this contextual change.

A couple of formal definitions are useful. First *a touches b* iff $ab \preceq aa$ and $a \neq b$. (I use both $\succeq$ and $\preceq$ in the standard way, as well as $\succ$ and $\prec$.) Second, *a is near b* iff $ab \preceq a_0b_0$ and a does not touch b.

A *weak geometrical structure of rigid bodies* $\mathbb{A} = (A, B, \succeq, a_0, b_0)$ satisfies the following axioms for all a, b, c, d, e and f in A.

A1. The set A is nonempty and finite.

A2. If $B(a, b, a)$ then $a = b$.

A3. If $B(a, b, c)$ then $B(c, b, a)$.

A4. If $B(a, b, c)$ and $B(b, d, c)$ then $B(a, b, d)$.

A5. If $ab \succeq cd$ and $cd \succeq ef$ then $ab \succeq ef$.

A6. $ab \succeq cd$ or $cd \succeq ab$.

A7. $ab \succeq ba$.

A8. $a_0b_0 \succ aa$.

A9. If $B(a, b, c)$ then $ac \succeq ab$.

The intuitive meaning of these weak axioms is, I think, obvious. That the present setup will not lead to a metric representation without substantial modification is apparent by the failure of the qualitative additivity axiom *If* $B(a, b, c)$, $B(a', b', c')$, $ab \succeq a'b'$ *and* $bc \succeq b'c'$ *then* $ac \succeq a'c'$.

A counterexample to this axiom is easily constructed by the case of body b' being much longer along the segment joining a' and c' than is b along the segment joining a and c. The most direct way out of this problem seems to be to introduce the qualitative length of a body along the line joining two other bodies. The notation $b(ac)$ could convey this idea, i.e., $b(ac)$ is the qualitative length of body b along the segment joining a and c. We would then add such axioms as *If* $b(ac) \succeq de$ *then* $a \neq c$, and $b(ac) \succ aa$. But there are other problems to be dealt with, so I have not attempted to go further in this lecture. For example, further counterexamples can easily be constructed to the modified additivity axiom given just above even with $b(ac) \succeq b'(a'c')$ added to the hypothesis. The simplest counterexample arises from consideration of concave bodies, but without further conditions convex bodies also can be used.

But the difficulties with weak geometrical structures of rigid bodies as axiomatized is more fundamental. They do not provide the proper setting for analyzing the spatial relation of nearness or other spatial relations. The automorphisms of such a structure do not provide anything like the intuitively correct answer. What is missing is a way of expressing the many potential spatial positions of two bodies a and b, all of which potential positions satisfy the relation of nearness. We need an analysis similar to that given for *in*. Let $\Phi(a, b; a_0, b_0)$ be the set of rigid motion transformations of a relative to b such that if $\varphi \in \Phi(a, b; a_0, b_0)$ then with respect to the nearness standard a_0, b_0, $\varphi(a)$ is near to b if and only if a is near to b. As before Φ is the symmetry group, but without additional geometrical assumptions, not much can be said about its structure. Moreover, without any restrictions a and b can be subject to an arbitrary rigid motion in physical space. But in many cases it is natural to restrict a and b to a given horizontal surface, such as a table top or the floor of a room. Then the set Φ is simplified by restricting the potential changes of relative position of a to that surface. This leads on to a familiar kind of result: by considering not just one but several nearness relations of a to a number of objects we restrict dramatically the symmetry group Φ, even in extreme cases, to the identity group. But it is also clear that such exact extreme results are not part of our natural concept of nearness.

Similar analyses can be given of the internal invariance properties of the other prepositions listed in Table 1. They all have different symmetry groups, but the exact nature of each group is determined by the particular shape and size of the relevant bodies, which may vary drastically from one context to another. It is not surprising that no computationally simple theory of invariance works for ordinary spatial relations as expressed in natural language, for the robust range of applicability of such relations is far too complex to have a simple uniform geometry.

Further applications. In addition to providing an analysis of spatial terms, the kind of approach outlined above can be used in a comparison of different languages. A natural question is whether there is a universal semantics of spatial perception or do different languages have intrinsically different semantics. Certainly there are subtle differences in the range of prepositions. Bowerman (1989) points out for example, that whereas we can say in English both *The cup is on the table* and *The fly is on the window*, in German we use the preposition *auf* for being on the table, but *an* for being on the vertical window.

Such differences are to be expected, although it is a matter of considerable interest to study how the prepositions of different languages overlap in their semantic coverage. It is a well-known fact that learning the correct use of prepositions is one of the most difficult aspects of learning a second language within the family of Indo-European languages.

One hope might be that there is a kind of universal spatial perception, so we might search for geometrical invariance across languages. But if we fit the geometries closely to individual languages, the primitives but not the theorems may be different in the sense of having a different geometrical meaning. A related set of questions can be asked about the order of developmental use of spatial prepositions by children. This seems to be a particularly good case for careful examination of what happens in different languages, because of the relatively concrete and definite rules of usage that govern spatial prepositions in each language in their literal use. Bowerman (1989) has an excellent discussion of many details, but does not focus on systematic geometrical aspects.

2 Geometry of Visual Space

Let me begin by reminding you of the classical alley experiments of Hillenbrand (1902) and Blumenfeld (1913). Almost all students of psychology at least know something about these classical experiments. I shall mainly refer to Blumenfeld's work because it was an improvement on that of Hillenbrand. As you will perhaps recall, Blumenfeld performed experiments with so-called parallel and equidistance alleys. The subject sits at a table in a darkened room. Looking straight ahead, he is asked to adjust two rows of points sources of light placed on either side of the normal plane, i.e., the vertical plane that bisects the horizontal segment joining the centers of the two eyes. The two furthest lights are fixed and are placed symmetrically and equidistant from the normal plane. In the case of the task being the construction of a parallel alley, the subject is asked to arrange the other lights so that they form two parallel lines extending toward him from the fixed lights. The subject's task is to arrange the lights so that they are perceived as lying on parallel lines in the subject's visual space. The other task is to construct an equidistance alley. In this experiment, all the lights except the two fixed lights are turned off and in sequence a pair of lights is presented, which are adjusted to be at the same perceptual distance apart as the fixed lights. Here the subject is making a clear judgment of equidistance, not of lines being parallel. When one pair of lights is turned off, another pair closer to the subject is presented for adjustment, etc. The important point now is that the physical configurations arising from the two experiments do not coincide, but in Euclidean geometry straight lines are parallel if and only if they are equidistant from each other along any mutual perpendiculars. Classically, the discrepancies observed in the Blumenfeld experiment are taken to be evidence that visual space is not Euclidean. The results are shown graphically and thereby most easily in Figure 1. In both the parallel alley and equidistance alley experiments, the lines are found to diverge as the adjusted pairs of points lie further away from the subject. But the angle of divergence tends to be greater in the case of parallel than in the case of equidistance alleys, as is clear in Figure 1. Since the most distant pair of points is the same for both alleys, this means that the equidistance alley lies outside the parallel alley. These results have been taken by Luneburg and others to support the hypothesis that visual space is hyperbolic, for this qualitative result is a property of hyperbolic space, even though there is some ambiguity in the fact that to a given line there is not a unique parallel line in hyperbolic space. Luneburg essentially used orthogonality to characterize being parallel, a matter that is discussed with some care in Indow (1979).

Although the experiments and Luneburg's conclusions are well known, the situation is not conceptually as clean as it might be for, as you will remember, the alternative to the space being Euclidean or hyperbolic is that it is elliptic—the restriction to these three choices will be

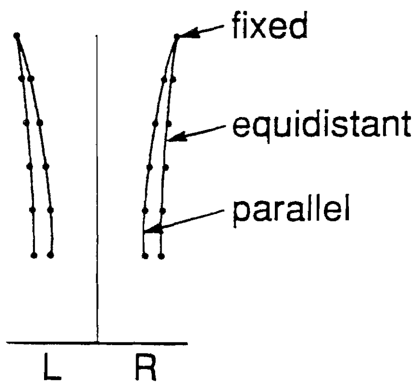

Figure 1. Diagram of classical alley experiments.

discussed more in a moment. However, no two lines can be parallel in elliptic space, so again a compromise must be struck in how the notion of parallel is to be handled. However, it must be said that for the local concept of two lines being parallel, it can be shown that in elliptic spaces the parallel alley lies outside the equidistance alley.

If matters were simply to be left with the classical alley experiments, which have been duplicated many times and have practically become a standard demonstration experiment, then we could settle the issue quickly, by concluding the evidence was excellent that if we must choose between the Riemannian surfaces of constant curvature in characterizing visual space, hyperbolic space is obviously the correct choice. However, as in most such matters in perception, the subsequent history of new and different experiments has ruled out any simple conclusion.

Before looking more systematically at some parametric experiments, it is important to note that the experiment that meets the criticisms given above of the notion of parallel is that of Blank (1961) who asked subjects to compare line segment bc and line segment ef in Figure 2. I am not describing the exact experimental protocol, but the point was to get judgments from subjects as to whether ef was half of bc (Euclidean space), less than half of bc (hyperbolic space), or more than half of bc (elliptic space). A majority, but not all, of the subjects supported the hyperbolic hypothesis. From a methodological standpoint this is in my judgment a very nice experiment, even though there are some natural qualms about judgments of one segment being twice another, as compared with a more direct qualitative judgment of equidistance or parallelness, or, to put the matter another way, in some fundamental geometrical sense the notion of parallel or equidistance is more fundamental than that of being half the length. This is a minor objection however, and I think it is overridden by the elegant way in which the problems of parallelness for hyperbolic and elliptic spaces are avoided.

Luneburg has been the central theorist of the hyperbolic conception of visual space and he has well worked out theoretical ideas that have led to a number of experiments (see for example his publications of 1947, 1948, and 1950). His central idea was to develop a parametric theory

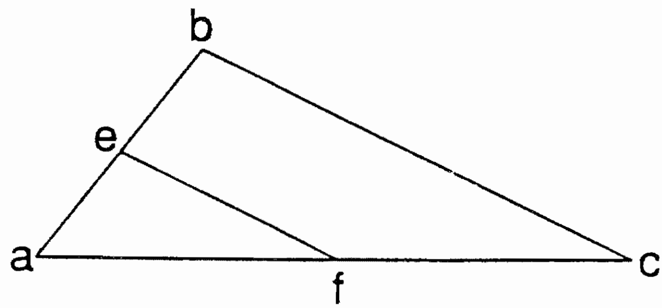

Figure 2. Illustration of comparisons required in Blank experiment.

based on the general assumption that in order to have free mobility of rigid bodies the space must be a Riemannian space of constant curvature. Luneburg used a somewhat unsatisfactory differential argument to get the space of constant curvature and did not refer in a detailed way to the classical Helmholtz-Lie space problem which concerns the characterization of spaces that satisfy free mobility. I shall not review that history here but only recall for purposes of present reference that the results came out as Luneburg had hoped. The only spaces tolerating free mobility of rigid bodies are Riemannian spaces of constant curvature if we demand satisfaction of certain other natural properties such as smoothness.

Luneburg showed that the line element ds can be expressed in terms of orthogonal sensory coordinates ξ, η and ζ by:

$$ds^2 = \frac{d\xi^2 + d\eta^2 + d\zeta^2}{[1 + \frac{1}{4}K(\xi^2 + \eta^2 + \zeta^2)]^2} \tag{1}$$

where for:

$$\begin{aligned} &\text{Euclidean space}: && K = 0, \\ &\text{hyperbolic space}: && K < 0, \\ &\text{elliptic space}: && K > 0. \end{aligned}$$

By introducing certain relatively natural but restrictive psychophysical assumptions, Luneburg shows that in physical coordinates Equation (1) becomes in polar coordinates

$$ds^2 = \frac{4}{(e^{\sigma\gamma} + Ke^{-\sigma\gamma})^2}(\sigma^2 d\gamma^2 + d\varphi^2 + \cos^2\varphi d\theta^2) \quad , \tag{2}$$

where on the basis of the psychophysical assumptions the relation to the sensory coordinates is postulated to be the following:

$$\begin{aligned} \xi &= 2e^{-\sigma}\cos\varphi\cos\theta \\ \eta &= 2e^{\sigma}\sin\varphi \\ \zeta &= 2e^{-\sigma}\cos\varphi\sin\theta. \end{aligned}$$

The parameter γ has a physical definition, but the parameters σ and K are estimated for each subject individually. The individual subject estimates for these two parameters, especially for K, is a reflection of the fact that the specific curvature observed by any two subjects, even if both observe hyperbolic space, will in general be different.

Many experiments have been done within this Luneburg framework. An early study is Hardy, et.al. (1953), but by far the most sustained experimental program has been that of Tarow Indow and his collaborators beginning in 1962 (Indow 1967, 1968, 1974, 1975, 1979, 1982; Indow, Inoue and Matsushima, 1962a, 1962b, 1963; Matsushima and Noguchi, 1967; and Nishikawa, 1967).

I shall not attempt to summarize these many carefully performed experiments in any detail. (For more detailed summary, see Suppes, et.al., 1989, pp. 145 - 153.) The most relevant three conclusions are these: (i) For most subjects $K < 0$, (ii) the estimates of K and σ were very unstable for many subjects even when the same experimental conditions were repeated, (iii) values of K and σ did not transfer well from one experimental setup to a different one. In particular, attempts to transfer K and σ from one set of experiments to the alley experiments did not work well at all.

The experiments I am summarizing in cursory form are in my judgment among the most careful of any psychological experiments involving parametric estimation that I can think of. The inference is rather about the unsatisfactory character of the theory rather than of the nature of the experiments. The great instability of the estimated parameters, especially for conceptual purposes the great instability of K, is in marked contrast to the precision with which the parameters of physical space are measured, with uniform values holding over a great range of circumstances. The instability and lack of generalizability naturally generate skepticism that it is the right scientific move to think of visual space in the same kinds of terms and within the kind of conceptual framework so common in examining the nature of physical space.

Moreover, there are several experiments that raise further doubts, beyond any question of instability of parameters, because they take the results for the nature of visual space outside the Luneburg framework. The first experiment to be mentioned is that of Wagner (1985). The methodological approach of this experiment is notable for two reasons. First, unlike the standard Luneburg experiments, the experiment was conducted outdoors in full daylight in a large field with subjects making judgments about the geometrical relations of 13 white stakes. Different procedures were used for measuring distances, angles, and areas. In particular magnitude estimation, category estimation, and constructing a simple scale map were used for judging distances, the only results to be considered here.

The results are extremely interesting and are contrary in important ways to essentially all of the Luneburg-type experiments. The important result is that there was spectacular foreshortening in depth perception. Let the x axis be the horizontal depth axis, that is, the axis perpendicular to the vertical plane through the eyes, and let the y axis be the horizontal frontal axis passing through the two eyes. Let two physical distances be such that $x = y$, that is, one distance is taken along the depth axis and the other along the frontal axis. Then in perceptual estimates (indicated by primes) of depth $x' = 0.5y'$ with of course some variation around 0.5 for individual subjects. The coefficient 0.5 is not some minor variation on standard physical Euclidean space but a major deviation in the form of an affine transformation of Euclidean space. It would be extremely interesting to determine if perceptual physics suffers such a large affine transformation as perceptual geometry. There have been other experiments reporting such depth foreshortening, for example, Battro, Netto, and Rozestraten (1976) but none as striking as the experiment I now turn to.

This is an elegant older experiment by Foley (1972) which leads to even stronger results, results that require the conception of visual space to lie outside any of the Riemannian spaces of constant curvature. In fact outside of any of the geometries ordinarily used in the study of perception. In Foley's 1972 experiment, the subject sat in a dark room with both eyes open, and a light *a* was fixed in the subject's visual field in the horizontal plane at the eye level. The subject was asked to set light *b* so that the line *ob* (*o* is the position of the subject) was perceptually perpendicular to line *ab*, and segment *ob* was apparently equal in length to segment *ab* (see Figure 3). Notice that this kind of task is very similar to the sort of task arising in the classical alley experiments, as far as the judgments required from the subject are concerned. The subject was next asked to set light *c* so that *oc* was perceived to be perpendicular to *ob* with *oc* equal in length to *ob*. The subject was then, as a final task, asked to judge the relative lengths of *oa* and *bc*. The important point is that for homogeneous Riemannian space, whether it be Euclidean, hyperbolic, or elliptic, by construction the right-angled isosceles triangles *oba* and *boc* should be congruent, and so *oa* and *bc* should be judged equal in length. The experimental results were quite different, however. Twenty-four subjects in 48 trials judged *bc* to be significantly longer than *oa*. It is important to note about Foley's experiment that this is not a question of various symmetrical. contextual features being present that lead to distortion. The general experimental environment is essentially that of the standard Luneburg experiments, or the standard alley experiments. The results however, are disastrous for any simple geometrical theory of visual space, for it requires us to move outside the framework of the standard elementary homogeneous spaces. I want to turn now to what these various results suggest for our study of visual space. I have organized the remarks under several different headings.

Study of Qualitative Axioms. Foley's experiment as well as other experiments he has conducted on the verification of Desargues' theorem, point toward a more intensive program of experimentally testing which individual qualitative axioms are satisfied. I say "which" with the understanding that such investigations could well begin with the standard classical primitives of geometry. For example, the linear relation of betweenness, standard qualitative relation of congruence, standard perpendicularity and parallel relations, etc. On the other hand, given the complicated and subtle nature of the results it may be that different primitive concepts will turn

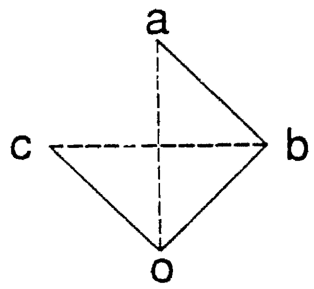

Figure 3. Illustration of comparisons required in Foley experiment.

out to have better invariant properties. For example, if it is hard to get coherent congruence results as we rotate 90 degrees from the frontal to the depth axis, it is natural to think that a different and more complex notion of congruence is needed than the standard one, which is independent of orientation—I return to this idea in a moment.

Space has not permitted me to say anything about projective geometry here, but the recent book of Cutting (1986) provides an excellent overview of projective questions that are natural to ask about visual space. The point of mentioning projective geometry here, however, is to remark that it may be that the kind of finite geometry characteristic of earlier work in projective geometry and still used for counterexamples and for other purposes, may turn out to be something that needs to be studied in the present context, for it may be that we would be able to satisfy certain qualitative axioms for a finite set of points as in the Foley experiments, but not their extension to a larger number. There is good reason why experimenters have been reluctant to move in this direction, for the possibilities of finding finite spaces that will satisfy a given fixed set of points are many, and it would be easy to get some extraordinarily ad hoc results which would not be of much interest. It seems to me, however, there are certain principled lines of inquiry that could be used and that might produce some very interesting conceptual results about the visual space of these particular experimental configurations that produce results that are so difficult to interpret. I also want to emphasize another way of thinking about these finite spaces that is different from the way finite projective spaces were thought about in the past. It seems to me the right way to think about them is in terms of geometrical *constructions*. In this case we can of course think of satisfying a much larger set of points by conceiving of the experimental configurations as being the first steps in constructing an ever larger configuration of points.

In the Foley experiments, for example, the subject has two fixed points (see Figure 3), namely points a and o. The other points are generated by construction. For the constructions involved, the line segments cut the depth axis and frontal axis all at a 45^o angle, and this could be a qualitative restriction on constructions leading to congruence. The observant reader will have

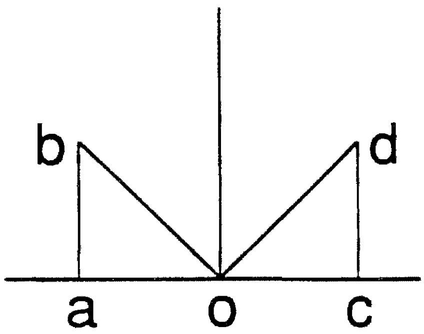

Figure 4. Illustration of comparisons permitted by restricted axiom of congruence.

noted that Foley's experiments satisfy the foreshortening result of Wagner's experiments. What Foley's results in conjunction with Wagner's show is that we cannot have some simple concept of congruence resulting from an affine transformation of the Euclidean space. We need something like a "directional" concept of congruence. Construction of a small number of points, with direction a part of the primitive concept, and also a part of the primitive concept of congruence, could lead to qualitative axioms producing both Foley's and Wagner's results.

Here is a sketch of one approach that might work. First, we use some standard qualitative primitives for affine spaces, e.g., betweenness or parallelness. Second, we develop in this framework affine congruence, i.e. congruence restricted to parallel segments. Third, add a restricted congruence axiom of the following sort that takes account of direction by requiring symmetry about the depth axis. (Here, as before o is the position of the observer.)

Axiom. If $B(a, o, c)$, $ao \simeq oc$, *ab and cd are parallel to the depth axis, and* $ab \simeq cd$, *then* $ob \simeq od$.

Note that the congruences postulated in the hypothesis of the axiom are affine, but the congruence of the conclusion is not. Figure 4 shows the simple construction. Other standard axioms of congruence that would be assumed are the following:

1. *If* $aa \simeq bc$ *then* $b = c$.

2. $ab \simeq ba$.

3. *If* $ab \simeq cd$ *and* $ab \simeq ef$, *then* $cd \simeq ef$.

4. *If b is between a and c, b′ is between a′ and c′,* $ab \simeq a'b'$ *and* $bc \simeq b'c'$, *then* $ac \simeq a'c'$.

The axioms of congruence given are not enough to prove a representation theorem for the kind of space suggested by the Foley and Wagner experiments, but they do provide a variety of testable consequences about congruence that are not falsified by the Foley and Wagner results.

Context Effects. Unfortunately, too energetic an effort to give a very detailed qualitative theory of the Foley and Wagner type of experiments could be misplaced, because already different results in a not too dissimilar arrangement are obtained in the classical alley experiments, which, at least in the obvious interpretation of the constructions made by the subjects in the experiments, do not satisfy the affine properties postulated for the Foley and Wagner experiments. This is perhaps one of the most disturbing aspects of experiments on visual space, namely, different experimental configurations can produce different geometrical results. I have made the point on several occasions that it may be the case that classical geometry is the wrong model for visual space. The kind of contextual effects to be seen in different arrangements is something much more characteristic of physics than geometry. In classical geometry there are no context effects. The properties of a configuration are not affected by properties of neighboring configurations. But in physics it is quite the opposite. When we study the interaction of two bodies we expect something very different to happen if we change the context by introducing a third body. Essentially every significant theory in physics has this kind of contextual property. What is disturbing, however, is the apparent difficulty of analyzing context in visual experiments in a way that would lead to interesting systematic results. To put the matter another way, in the framework of a central thrust of this lecture, we have not yet been successful in finding general principles of visual perception that have the appropriate invariance properties. It is in fact an open question whether satisfactory general principles exist. Our perceptual apparatus may be a pluralistic assembly of systems without strong unifying principles.

REFERENCES

Aristotle, *On the heavens*, Translated by W.K.G. Guthrie (1939). Cambridge, MA. Harvard University Press.

Battro, A.M., Netto, S.P., & Rozestraten, R.J.A. (1976). Riemannian geometries of variable curvature in visual space: Visual alleys, horopters, and triangles in big open fields. *Perception, 5*, 9-23.

Blank, A.A., (1961). Curvature of binocular visual space. An experiment. *Journal of the Optical Society of America, 51*, 335-339.

Blumenfeld, W. (1913). Untersuchungen über die scheinbare Grösse in Sehraume. *Zeitschrift für Psychologie und Physiologie der Sinnesorgane, 65*, 241-404.

Bowerman, M. (1989). Learning a semantic system. What role do cognitive predispositions play? In M.L. Rice & R.L. Schiefelbusch (Eds.), *The Teachability of Language* (pp. 133-169). Baltimore: Paul H. Brookes.

Crangle, C. & Suppes, P. (1989). Geometrical semantics for spatial prepositions. In P.A. French, T.E. Uehling, Jr., and Howard K. Wettstein (Eds.), *Midwest Studies in Philosophy, Volume XIV* (pp. 399-422). Notre Dame, Indiana: Univ. of Notre Dame Press.

Cutting, J.E. (1986). *Perception with an eye for motion.* Cambridge, MA: The MIT Press.

Foley, J.M. (1972). The size-distance relation and intrinsic geometry of visual space: Implications for processing. *Vision Research, 13*, 323-332.

Hardy, L.H., Rand, G., Rittler, M.C., Blank, A.A., & Boeder, P. (1953). *The geometry of binocular space perception.* Knapp Memorial Laboratories, Institute of Ophthalmology, Columbia University College of Physicians and Surgeons.

Hillenbrand, F. (1902). Theorie der scheinbaren Grösse bei binocularem Sehen. *Denkschriften d. Wiener Akademie d. Wissenschaften. Mathematisch-Naturwissenschaftliche Classe. 7*, 255-307.

Indow, T. (1967). Two interpretations of binocular visual space: Hyperbolic and Euclidean. *Annals of the Japan Association for Philosophy of Science, 3*, 51-64.

Indow, T. (1968). Multidimensional mapping of visual space with real and simulated stars. *Perception & Psychophysics, 3*, 45-53.

Indow, T. (1974). Applications of multidimensional scaling in perception. In *Handbook of perception. Vol. 2, Psychophysical judgment and measurement* (pp. 493-531). New York: Academic Press.

Indow, T. (1975). An application of MDS to study of binocular visual space. U.S.-Japan Seminar: *Theory, methods and applications of multidimensional scaling and related techniques.* University of California, August 20-24, San Diego, Calif.

Indow, T. (1979). Alleys in visual space. *Journal of Mathematical Psychology, 19*, 221-258.

Indow, T. (1982). An approach to geometry of visual space with no a priori mapping functions: Multidimensional mapping according to Riemannian metrics. *Journal of Mathematical Psychology, 26*, 204-236.

Indow, T., Inoue, E., & Matsushima K. (1962a). An experimental study of the Luneburg theory of binocular space perception (1). The 3- and 4-point experiments. *Japanese Psychological Research, 4*, 6-16.

Indow, T., Inoue, E., & Matsushima K. (1962b). An experimental study of the Luneburg theory of binocular space perception (2). The alley experiments. *Japanese Psychological Research, 4*, 17-24.

Indow, T., Inoue, E., & Matsushima K. (1963). An experimental study of the Luneburg theory of binocular space perception (3). The experiment in a spacious field. *Japanese Psychological Research, 5*, 10-27.

Levelt, W.J.M. (1982). Cognitive styles in the use of spatial direction terms. In R.J. Jarvella & W. Klein (Eds.), *Speech, place, and action*, (pp. 251-268). John Wiley & Sons Ltd.

Levelt, W.J.M. (1984). Some perceptual limitations on talking about space. In A.J. Van Doorn, W.A. van de Grind, & J. Koenderink (Eds.), *Limits in perception*, (pp. 323-358). Utrecht: VNU Press.

Luneburg, R.K. (1947). *Mathematical analysis of binocular vision.* Princeton, NJ: Princeton University Press.

Luneburg, R.K. (1948). Metric methods in binocular visual perception. In *Studies and essays*, (pp. 215-240). New York: Interscience.

Luneburg, R.K. (1950). The metric of binocular visual space. *Journal of the Optical Society of America, 40*, 627-642.

Matsushima, K., & Noguchi, H. (1967). Multidimensional representation of binocular visual space. *Japanese Psychological Research, 9*, 83-94.

Nishikawa, Y., (1967). Euclidean interpretation of binocular visual space. *Japanese Psychological Research, 9*, 191-198.

Ptolemy, C. (1984). *The almagest.* Translated by G. J. Toomer. New York: Springer Verlag.

Suppes, P. (1959). Axioms for relativistic kinetics with or without parity. In L. Henkin, P. Suppes, & A. Tarski (Eds.), *The axiomatic method with special reference to geometry and physics.* (Proceedings of an international symposium held at the University of California, Berkeley, December 16, 1957 - January 4, 1958, pp. 291-307.) Amsterdam: North-Holland.

Suppes, P. (1972). Some open problems in the philosophy of space and time. *Synthese, 24*, 298-316.

Suppes, P., Krantz, D.H., Luce, R.D., & Tversky, A. (1989). *Foundations of measurement, Volume II, Geometrical, threshold, and probabilistic representations.* San Diego, Calif.: Academic Press, Inc.

Toth, L. F. (1964). *Regular figures.* New York: Macmillan.

Wagner, M. (1985). The metric of visual space. *Perception & Psychophysics, 38*, 483-495.