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THE PROBABILISTIC ARGUMENT FOR A NON-CLASSICAL LOGIC OF QUANTUM MECHANICS*

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The aim of this paper is to state the single most powerful argument for use of a non-classical logic in quantum mechanics. In outline the argument is the following. The working logic of a science is the logic of the events and propositions to which probabilities are assigned. A probability should be assigned to every element of the algebra of events. In the case of quantum mechanics probabilities may be assigned to events but not, without restriction, to the conjunction of two events. The conclusion is that the working logic of quantum mechanics is not classical. The nature of the logic that is appropriate for quantum mechanics is examined.

1. The argument. The aim of this paper is simple. I want to state as clearly as possible, without a long discursion into technical questions, what I consider to be the single most powerful argument for use of a non-classical logic in quantum mechanics. There is a very large mathematical and philosophical literature on the logic of quantum mechanics, but almost without exception, this literature provides a very poor intuitive justification for considering a non-classical logic in the first place. A classical example in the mathematical literature is the famous article by Birkhoff and von Neumann [1]. Although Birkhoff and von Neumann pursue in depth development of properties of lattices and projective geometries that are relevant to the logic of quantum mechanics, they devote less than a third of a page (p. 831) to the physical reasons for considering such lattices. Moreover, the few lines they do devote are far from clear. The philosophical literature is just as bad on this point. One of the better known philosophical discussions on these matters is that found in the last chapter of Reichenbach's book [3] on the foundations of quantum mechanics. Reichenbach offers a three-valued truth-functional logic which seems to have little relevance to quantum-mechanical statements of either a theoretical or experimental nature. What Reichenbach particularly fails to show is how the three-valued logic he proposes has any functional role in the theoretical development of quantum mechanics. It is in fact fairly easy to show that the logic he proposes could not possibly be adequate for a systematic theoretical statement of the theory as it is ordinarily conceived. The reasons for this will become clear later on in the present paper.

The main premises of the argument I outline in this paper are few in number. I state them at this point without detailed justification in order to give the broad outline of the argument the simplest possible form.

Premise 1. *In physical or empirical contexts involving the application of probability theory as a mathematical discipline, the functional or working logic of importance is the logic of the events or propositions to which probability is assigned, not the logic of qualitative or intuitive statements to be made about the mathematically formulated theory.* (In the classical applications of probability theory, this logic of events is a Boolean algebra

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of sets; for technical reasons that are unimportant here this Boolean algebra is usually assumed to be countably additive, i.e., a σ-algebra.)

Premise 2. *The algebra of events should satisfy the requirement that a probability is assigned to every event or element of the algebra.*

Premise 3. *In the case of quantum mechanics probabilities may be assigned to events such as position in a certain region or momentum within given limits, but the probability of the conjunction of two such events does not necessarily exist.*

Conclusion: *The functional or working logic of quantum mechanics is not classical.*

From a scientific standpoint the conclusion from the premises is weak. All that is asserted is that the functional logic of quantum mechanics is not classical, which means that the algebra of events is not a Boolean algebra. Nothing is said about what the logic of quantum mechanics *is*. That question will be considered shortly. First I want to make certain that the support for the premises stated is clear, as well as the argument leading from the premises to the conclusion.

Concerning the first premise, the arguments in support of it are several. A source of considerable confusion in the discussion of the logic of quantum mechanics has been characterization of the class of statements whose logic is being discussed. On the one hand we are presented with the phenomenon that quantum mechanics is a branch of physics that uses highly developed mathematical tools, and on the other hand, discussions of logic deal with the foundations of mathematics itself. It is usually difficult to see the relation between characterization of the sentential connectives that seem appropriate for a new logic and the many mathematical concepts of an advanced character that must be available for actual work in quantum mechanics. The problem has often been posed as how can one consider changing the logic of quantum mechanics when the mathematics used in quantum mechanics depends in such a thorough fashion on classical logic. The point of this first premise is to narrow and sharpen the focus of the discussion of the logic of an empirical science. As in the case of quantum mechanics, we shall take it for granted that probability theory is involved in the mathematical statement of the theory. In every such case a logic of events is required as an underpinning for the probability theory. The structure of the algebra of events expresses in an exact way the logical structure of the theory itself.

Concerning the second premise the arguments for insisting that a probability may be assigned to every event in the algebra is already a part of classical probability theory. It is only for this reason that one considers an algebra, or σ-algebra, of sets as the basis for classical probability theory. If it were permitted to have events to which probabilities could not be attached, then we could always take as the appropriate algebra the set of all subsets of the basic sample space. The doctrine that the algebra of events must have the property asserted in the second premise is too deeply embedded in classical probability theory to need additional argument here. One may say that the whole point of making explicit the algebra of events is just to make explicit those sets to which probabilities may indeed be assigned. It would make no sense to have an algebra of events that was not the entire family of subsets of the given sample space and yet not be able to assign a probability to each event in the algebra.

Concerning the third premise it is straightforward to show that the algebra of events in quantum mechanics cannot be closed under conjunction or intersection of events. The event of a particle's being in a certain region of space is well-defined in all treatments of classical quantum mechanics. The same is true of the event of the particle's momentum's being in a certain region as well. If the algebra of events were a Boolean algebra we could then ask at once for the probability of the event consisting of the conjunction of the first two, that is, the event of the particle's being in a certain region at a given time t and also having its momentum lying in a certain interval at the same time t. What may be shown is that the probability of such a joint event does not exist in the classical theory. The argument goes back to Wigner [7], and I have tried to make it in as simple and direct a fashion as possible in Suppes [4]. The detailed argument shall not be repeated here. Its main line of development is completely straightforward. In the standard formalism, we may compute the expectation of an operator when the quantum-mechanical system is in a given state. In the present case the operator we choose is the usual one for obtaining the characteristic function of a probability distribution of two variables. Having obtained the characteristic function we then invert it by the usual Fourier methods. Inversion should yield the density corresponding to the joint probability distribution of position and momentum. It turns out that for most states of any quantum-mechanical system the resulting density function is not the density function of any genuine joint probability distribution. We conclude that in general the joint distribution of two random variables like position and momentum does not exist in quantum mechanics and, consequently, we cannot talk about the conjunction of two events defined in terms of these two random variables. From the standpoint of the logic of science, the fundamental character of this result is at a much deeper level than the uncertainty principle itself, for there is nothing in the uncertainty principle as ordinarily formulated that runs counter to classical probability theory.

The inference from the three premises to the conclusion is straightforward enough hardly to need comment. From premise (1) we infer that the functional logic of events is the formal algebra of events on which a probability measure is defined. According to premise (2) every element, i.e., event, of the algebra must be assigned a probability. According to premise (3) the algebra of events in quantum mechanics cannot be closed under the conjunction of events and satisfy premise (2). Hence the algebra of events in quantum mechanics is not a Boolean algebra, because every Boolean algebra is closed under conjunction. Whence according to premise (1) the functional logic of quantum mechanics is not a Boolean algebra and thus is not classical.

2. The logic. Although the conclusion of the argument was just the negative statement that the logic of quantum mechanics is not classical, a great deal more can be said on the positive side about the sort of logic that does seem appropriate. To begin with it will be useful to record the familiar definition of an algebra, and σ-algebra, of sets.

Definition 1. *Let X be a non-empty set. $\mathscr{F}$ is a classical algebra of sets on X if and only if $\mathscr{F}$ is a non-empty family of subsets of X and for every A and B in $\mathscr{F}$:*

1. $\sim A \in \mathscr{F}$.
2. $A \cup B \in \mathscr{F}$;

Moreover, if $\mathscr{F}$ is closed under countable unions, that is, if for $A_1, A_2, \ldots, A_n, \ldots \in \mathscr{F}$,

$$\bigcup_{i=1}^{\infty} A_i \in \mathscr{F},$$

then $\mathscr{F}$ is a classical σ-algebra on X.

It is then standard to use the concepts of Definition 1 in defining the concept of a classical probability space. In this definition we assume that the set-theoretical structure of X, $\mathscr{F}$ and P is familiar; in particular, that X is a non-empty set, $\mathscr{F}$ a family of subsets of X and P a real-valued function defined on $\mathscr{F}$.

Definition 2. *A structure $\mathscr{X} = \langle X, \mathscr{F}, P\rangle$ is a finitely additive classical probability space if and only if for every A and B in $\mathscr{F}$:*

P1. *$\mathscr{F}$ is a classical algebra of sets on X;*
P2. *$P(A) \geqslant 0$;*
P3. *$P(X) = 1$;*
P4. *If $A \cap B = 0$, then $P(A \cup B) = P(A) + P(B)$.*

Moreover, $\mathscr{X}$ is a classical probability space (without restriction to finite additivity) if the following two axioms are also satisfied:

P5. *$\mathscr{F}$ is a σ-algebra of sets on $\mathscr{X}$;*
P6. *If $A_1, A_2, \ldots,$ is a sequence of pairwise incompatible events in $\mathscr{F}$*, i.e., $A_i \cap A_j = 0$ *for $i \neq j$, then*

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

In modifying the classical structures characterized in Definitions 1 and 2 to account for the truculent "facts" of quantum mechanics, there are a few relatively arbitrary choice points. One of them needs to be described in order to explain an aspect of the structures soon to be defined. I pointed out earlier that the joint probability of two events does not necessarily exist in quantum mechanics. A more particular question concerns the joint probability of two disjoint events. In this case there is no possibility of observing both of them, since the very structure of the algebra of events rules this out. On the other hand, it is theoretically convenient to include the union of two such events in the algebra of sets, or a denumerable sequence of pairwise disjoint events, in the case of a σ-algebra. This liberal attitude toward the concept of event has been adopted here, but it should be noted that it would be possible to take a stricter attitude without affecting the concept of an observable in any important way. (This stricter attitude is taken by Kochen and Specker [2], but they also deliberately exclude all probability questions in their consideration of the logic of quantum mechanics.)

So the logic of quantum mechanics developed here permits the union of disjoint events apart from any question of noncommuting random variables' being involved in their definition. A more detailed discussion of this point may be found in Suppes [5]. Roughly speaking, the definitions that follow express the idea that the probability distribution of a single quantum-mechanical random variable is classical, and the deviations arise only when several random variables or different kinds of events are considered.

The approach embodied in Definition 3 follows Varadarajan [6]; it differs in that Varadarajan does not consider an algebra of sets, but only the abstract algebra.

Definition 3. *Let X be a non-empty set. $\mathscr{F}$ is a quantum-mechanical algebra of sets on X if and only if $\mathscr{F}$ is a non-empty family of subsets of X and for every A and B in $\mathscr{F}$:*

1. $\sim A \in \mathscr{F}$;
2. *If* $A \cap B = 0$ *then* $A \cup B \in \mathscr{F}$;

Moreover, if $\mathscr{F}$ *is closed under countable unions of pairwise disjoint sets, that is, if* A_1, A_2,... *is a sequence of elements of* $\mathscr{F}$ *such that for* $i \neq j$, $A_i \cap A_j = 0$

$$\bigcup_{i=1}^{\infty} A_i \in \mathscr{F},$$

then $\mathscr{F}$ *is a quantum-mechanical* σ*-algebra of sets.*

The following elementary theorem is trivial.

Theorem 1. If $\mathscr{F}$ *is a classical algebra (or* σ*-algebra) of sets on* X *then* $\mathscr{F}$ *is also a quantum-mechanical algebra (or* σ*-algebra) of sets on* X.

The significance of Theorem 1 is apparent. It shows that the concept of a quantum-mechanical algebra of sets is a strictly weaker concept than that of a classical algebra of sets. This is not surprising in view of the breakdown of joint probability distributions in quantum mechanics. We cannot expect to say as much, and the underlying logical structure of our probability spaces reflects this restriction.

It is hardly necessary to repeat the definition of probability spaces, because the only thing that changes is the condition on the algebra $\mathscr{F}$, but in the interest of completeness and explicitness it shall be given.

Definition 4. *A structure* $\mathscr{X} = \langle X, \mathscr{F}, P\rangle$ *is a finitely additive quantum-mechanical probability space if and only if for every* A *and* B *in* $\mathscr{F}$:

P1. $\mathscr{F}$ *is a quantum-mechanical algebra of sets on* X;
P2. $P(A) \geqslant 0$;
P3. $P(X) = 1$;
P4. *If* $A \cap B = 0$, then $P(A \cup B) = P(A) + P(B)$.

Moreover, X *is a quantum-mechanical probability space (without restriction to finite additivity) if the following two axioms are also satisfied:*

P5. $\mathscr{F}$ *is a quantum-mechanical* σ*-algebra of sets on* X;
P6. *If* $A_1, A_2, \ldots,$ *is a sequence of pairwise incompatible events in* $\mathscr{F}$, i.e., $A_i \cap A_j = 0$ *for* $i = j$, *then*

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

It is evident from the close similarity between Definitions 2 and 4 that we have as an immediate consequence of Theorem 1 the following result:

Theorem 2. Every classical probability space is also a quantum-mechanical probability space.

It goes without saying that in the case of both of these theorems it is easy to give counterexamples to show that their converses do not hold.

Quantum-mechanical probability spaces can be used as the basis for an axiomatic development of classical quantum mechanics, but the restriction to algebras of sets in order to stress the analogy to classical probability spaces is too severe. The spaces defined are adequate for developing the theory of all observables that may be defined in terms of position and momentum, but not for the more general theory. The fundamental characteristic of the general theory is that not every quantum-mechanical algebra may be embedded in a Boolean algebra, and thus is not isomorphic to a quantum-mechanical algebra of sets, because every such algebra of sets is obviously embeddable in the Boolean algebra of the set of all subsets of X.

It is thus natural to consider the abstract analogue of Definition 3 and define the general concept of a quantum-mechanical algebra. (The axioms given here simplify those in Suppes [5], which are in turn based on Varadarajan [6].) Let A be a non-empty set, corresponding to the family $\mathscr{F}$ of Definition 2, let $\leqslant$ be a binary relation on A— the relation $\leqslant$ is the abstract analogue of set inclusion, let $'$ be a unary operation on A— the operation $'$ is the abstract analogue of set complementation, and let 1 be an element of A— the element 1 is the abstract analogue of the sample space X. We then have:

Definition 5. *A structure $\mathfrak{A} = \langle A, \leqslant, ', 1\rangle$ is a quantum-mechanical algebra if and only if the following axioms are satisfied for every a, b and c in A:*

1. $a \leqslant a$;
2. *If* $a \leqslant b$ *and* $b \leqslant a$ *then* $a = b$;
3. *If* $a \leqslant b$ *and* $b \leqslant c$ *then* $a \leqslant c$;
4. *If* $a \leqslant b$ *then* $b' \leqslant a'$;
5. $(a')' = a$;
6. $a \leqslant 1$;
7. *If* $a \leqslant b$ *and* $a' \leqslant b$ *then* $b = 1$;
8. *If* $a \leqslant b'$ *then there is a* c *in* A *such that* $a \leqslant c$, $b \leqslant c$, *and for all* d *in* A *if* $a \leqslant d$ *and* $b \leqslant d$ *then* $c \leqslant d$;
9. *If* $a \leqslant b$ *then there is a* c *in* A *such that* $c \leqslant a'$, $c \leqslant b$ *and for every* d *in* A *if* $a \leqslant d$ *and* $c \leqslant d$ *then* $b \leqslant d$.

The only axioms of any complexity are the last three. If the operation of addition for disjoint elements were given the three axioms would be formulated as follows:

7! $a + a' = 1$;
8! $a + b$ *is in* A;
9! *If* $a \leqslant b$ *then there is a* c *in* A *such that* $a + c = b$.

The difficulty with the operation of addition is that we do not want it to be defined except for disjoint elements, i.e., elements a and b of A such that $a \leqslant b'$.

It should also be apparent that we obtain a σ-algebra by adding to the axioms of Definition 5 the condition that for any sequence of pairwise disjoint elements $a_1, a_2, \ldots, a_n, \ldots$ of A there is a c in A such that for all n, $a_n \leqslant c$ and for every d in A, if for every n, $a_n \leqslant d$, then $c \leqslant d$.

Although it may be apparent, in the interest of explicitness, it is desirable to prove the following theorem.

Theorem 3. Every quantum-mechanical algebra of sets is a quantum-mechanical algebra in the sense of Definition 5.

Proof: Let $\mathscr{F}$ be a quantum-mechanical algebra of sets on X. The relation $\leq$ of Definition 5 is interpreted as set inclusion $\subseteq$, and Axioms 1-3 immediately hold. The complementation is interpreted as set complementation with respect to X, and Axioms 4 and 5 hold in this interpretation. The Unit 1 is interpreted as the set X, and Axiom 6 holds because for any A in $\mathscr{F}$, $A \subseteq X$. In these case of Axiom 7 it is evident from elementary set theory that if $A \subseteq B$ and $\sim A \subseteq B$, then $A \cup \sim A \subseteq B$, whence $X \subseteq B$, but $B \subseteq X$, and so $B = X$. Regarding Axiom 8, if $A \subseteq \sim B$ then $A \cap B = 0$, so $A \cup B \in \mathscr{F}$ by virtue of the second axiom for algebras of sets, and we may take $C = A \cup B$ to satisfy the existential requirement of the axiom, because $A \subseteq A \cup B$, $B \subseteq A \cup B$, and if $A \subseteq D$ and $B \subseteq D$ then $A \cup B \subseteq D$. Finally, as to Axiom 9, if $A \subseteq B$ then we first want to show that $B \sim A \in \mathscr{F}$. By hypothesis $A, B \in \mathscr{F}$, whence $\sim B \in \mathscr{F}$, and since $A \subseteq B$, $A \cap \sim B = 0$ and thus $A \cup \sim B \in \mathscr{F}$, but then because $\mathscr{F}$ is closed under complementation, $\sim (A \cup \sim B) = \sim A \cap B = B \sim A \in \mathscr{F}$, as desired. It is easily checked, in order to verify Axiom 9 that because $A \subseteq B$, we have $B \sim A \subseteq \sim A$, $B \sim A \subseteq B$ and for every set D in $\mathscr{F}$, if $A \subseteq D$ and $B \sim A \subseteq D$ then $B \subseteq D$, since $A \cup (B \sim A) \subseteq D$ and $A \cup (B \sim A) = B$. Thus $B \sim A$ is the desired C, which completes the proof.

To obtain a sentential calculus for quantum-mechanical algebras, we define the notion of validity in the standard way. More particularly, in the calculus implication $\rightarrow$ corresponds to the relation $\leqslant$ and negation $\neg$ to the complementation operation $'$. We say that a sentential formula is quantum-mechanically valid if it is satisfied in all quantum-mechanical algebras, i.e., if under the expected interpretation the formula designates the element *1* of the algebra. The set of such valid sentential formulas characterizes the sentential logic of quantum mechanics. The axiomatic structure of this logic will be investigated in a subsequent paper.

I conclude with a brief remark about Reichenbach's three-valued logic. It is easy to show that the quantum-mechanical logic defined here is not truth-functional in his three values (for more details see Suppes [5]). It seems clear to me that his three-valued logic has little if anything to do with the underlying logic required for quantum-mechanical probability spaces, and I have tried to show why the logic of quantum-mechanical probability is *the* logic of quantum mechanics. What I have not been able to do within the confines of this paper is to make clear precisely why the algebras characterized in Definition 5 are exactly appropriate to express the logic of quantum-mechanical probability. The argument in support of this choice is necessarily rather long and technical. A fairly good case is made out in detail in Varadarajan [6].

However, apart from giving a mathematically complete argument for Definition 5, it may be seen that quantum-mechanical algebras have many intuitive properties in common with Boolean or classical algebras. The relation of implication or inclusion has most of its ordinary properties, the algebras are closed under negation, and the classical law of double negation holds. What is lacking are just the properties of closure under union and intersection—or disjunction and conjunction—that would cause difficulties for non-existent joint probability distributions.

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